# Electrical and Mechanical Passive Network Synthesis

Michael Z.Q. Chen[1,2] and Malcolm C. Smith[1]

[1] Department of Engineering, University of Cambridge, U.K.
zc214@cam.ac.uk/mzqchen@gmail.com, mcs@eng.cam.ac.uk
[2] Department of Engineering, University of Leicester, U.K.

**Summary.** The context of this paper is the application of electrical circuit synthesis to problems of mechanical control. The use of the electrical-mechanical analogy and the inerter mechanical element is briefly reviewed. Classical results from passive network synthesis are surveyed including Brune's synthesis, Bott-Duffin's procedure, Darlington's synthesis, minimum reactance extraction and the synthesis of biquadratic functions. New results are presented on the synthesis of biquadratic functions which are realisable using two reactive elements and no transformers.

## 1 Introduction

Passive network synthesis is a classical subject in electrical circuit theory which experienced a "golden era" from the 1930s through to the 1960s. Renewed interest in this subject has recently arisen due to the introduction of a new two-terminal element called the *inerter* and the possibility to directly exploit electrical synthesis results for mechanical control [38]. Applications of this approach to vehicle suspension [39,30], control of motorcycle steering instabilities [19,20] and vibration absorption [38] have been identified.

Despite the relative maturity of the field, there are aspects of passive network synthesis which can be considered as incomplete. For example, the question of minimality of realisation in terms of the total number of elements used is far from solved. For mechanical networks, efficiency of realisation is much more important than for electrical networks. Also, for mechanical networks it is often desirable that no transformers are employed, due to the fact that levers with unrestricted ratios can be awkward to implement. However, the only general method for transformerless electrical synthesis—the method of Bott and Duffin [7] and its variants [29,31,40,21]—appears to be highly non-minimal.

The purpose of this paper is to review some of the background electrical circuit synthesis theory and present some new results on the transformerless synthesis of a sub-class of biquadratic functions.

## 2 The Electrical and Mechanical Analogy

The principal motivation for the introduction of the inerter in [38] was the synthesis of passive mechanical networks. It was pointed out that the standard

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form of the electrical-mechanical correspondences (where the spring, mass and damper are analogous to the inductor, capacitor and resistor) was restrictive for this purpose, because the mass element effectively has one terminal connected to ground. To allow the full power of electrical circuit synthesis theory to be translated over to mechanical networks, it is necessary to replace the mass element by a genuine two-terminal element with the property that the (equal and opposite) force applied at the terminals is proportional to the *relative* acceleration between them. In the notation of Fig. 1, the inerter obeys the force-velocity law $F = b(\dot{v}_1 - \dot{v}_2)$, where the constant of proportionality $b$ is called the inertance and has the units of kilograms and $v_1$, $v_2$ are the velocities of the two terminals with $v = v_1 - v_2$. Fig. 2 shows the new table of element correspondences in the force-current analogy where force and current are the "through" variables and velocity and voltage are the "across" variables. The admittance $Y(s)$ is the ratio of through to across quantities, where $s$ is the standard Laplace transform variable.

The mechanical realisation of an inerter can be achieved using a flywheel that is driven by a rack and pinion, and gears (see Fig. 3). The value of the inertance $b$ is easy to compute in terms of the various gear ratios and the flywheel's moment

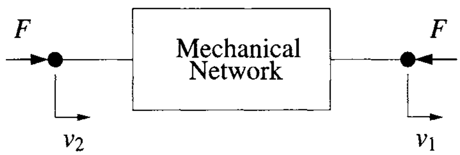


**Fig. 1.** Free-body diagram of a two-terminal mechanical element with force-velocity pair $(F, v)$

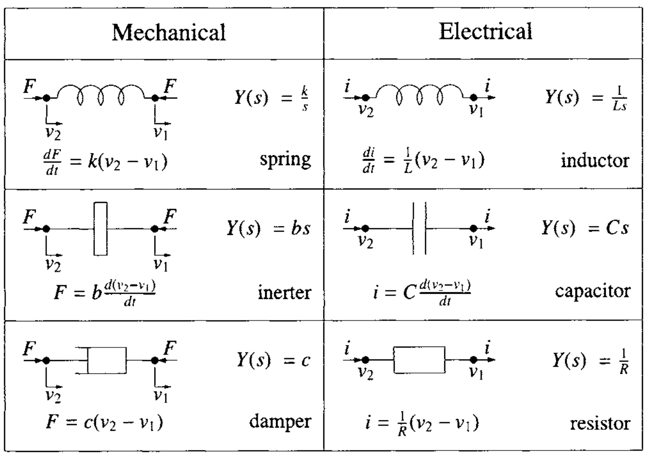


**Fig. 2.** Circuit symbols and correspondences with defining equations and admittance $Y(s)$

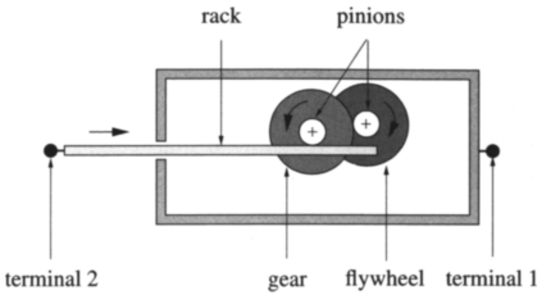


**Fig. 3.** Schematic of a mechanical model of an inerter

of inertia [38]. In general, if the device gives rise to a flywheel rotation of $\alpha$ radians per metre of relative displacement between the terminals, then the inertance of the device is given by $b = J\alpha^2$ where $J$ is the flywheel's moment of inertia. Other methods of construction are described in [37].

With the new correspondence in Fig. 2, synthesis methods from electrical networks can be directly translated over to the mechanical case. In particular, Bott-Duffin's synthesis result [7] provides a means to realise an arbitrary positive-real mechanical admittance or impedance with networks comprising springs, dampers and inerters [38]. We will review some of the basic results from network synthesis in the next section.

## 3 Passive Network Synthesis

From the vast amount of literature which has been devoted to electrical circuit synthesis, we now highlight some of the fundamental results which are relevant for our application. Interesting overviews on the history of passive network synthesis can be found in [6] and [17]. Readers are referred to [1,4,5,27,28,43,47] for a more detailed treatment on the subject.

By the 1920s, researchers had started searching for a systematic way of realising passive networks. One of the early contributions is Foster's reactance theorem [22] and it is often described as the first devoted to synthesis of networks in the modern sense. The theorem states a necessary and sufficient condition for an impedance function to be realisable as the driving point impedance of a lossless one-port. The first paper dealing explicitly with the realisation of a one-port with the impedance being a prescribed function of frequency is Cauer's 1926 contribution, based on continuous fraction expansions [6,9]. With Cauer's and Foster's theorems, the synthesis problem for one-ports containing two kinds of elements only was solved.

In Brune's ground-breaking work [8], the class of positive-real functions was introduced. A rational function $Z(s)$ is defined to be *positive-real* if (i) $Z(s)$ is analytic in $\mathrm{Re}[s] > 0$ and (ii) $\mathrm{Re}[Z(s)] \geq 0$ for all $s$ with $\mathrm{Re}[s] > 0$. He showed that there is a fundamental correspondence between positive-real functions and passive electrical circuits. In particular he showed that: (1) the

driving-point impedance or admittance of any linear two-terminal (one-port) network is positive-real, and conversely, (2) given *any* positive-real function, a two-terminal network comprising resistors, capacitors, inductors and transformers can be found which has the given function as its driving-point impedance or admittance. Brune's construction begins with the *Foster preamble* which reduces the positive-real function to a "minimum function", which is a positive-real function that has no poles or zeros on the imaginary axis or infinity and has a real part that vanishes for at least one finite real frequency. The next part of the construction is the "Brune cycle" which expresses the minimum function as a lossless coupling network connected to a positive-real function of strictly lower degree. The whole process is then repeated until a degree zero function (resistor) is reached.

For a number of years following Brune's paper, it was thought that the transformers appearing in the synthesis of general positive-real functions were unavoidable. It was therefore a surprise when a realisation procedure was published by Bott and Duffin which does not require transformers [7]. Similar to Brune's procedure, Bott-Duffin's approach also starts with the Foster preamble to reduce the positive-real function to a minimum function. It then makes use of the Richard's theorem [33], which is a generalisation of Schwarz's lemma [26], to express the minimum function as a lossless coupling network connected to *two* positive-real functions of strictly lower degree. Thus the procedure gives the appearance of being wasteful in terms of the number of components required. How wasteful it is remains an open question.

Since 1949 the only general simplifications of Bott-Duffin's method are just variants of the procedure, e.g. Pantell's procedure [29], Reza's procedure [31] and Storer's procedure [40]. All three variants work by unbalancing the bridge configuration within the lossless coupling network in Bott-Duffin's realisation to reduce the number of elements in the network from six to five in each cycle. Later, Fialkow and Gerst independently proved a similar result [21].

An important alternative proof of Brune's theorem was obtained in 1939 by Darlington [16]. The realisation method expressed the positive-real function as a lossless two-port terminated in a single resistor. The lossless two-port was realised using transformers as well as inductors and capacitors. The method was also called "minimum resistance synthesis". Connections of the method with classical interpolation were later identified [18] which have served to set the method in a general context.

A different set of techniques for passive network synthesis was based on a state-space formulation [1]. One of the central ideas is "reactance extraction" in which the impedance is represented as a multi-port with $n$ of the ports terminated by inductors or capacitors, where $n$ is the McMillan degree of the transfer-function. Central to the approach is the "positive-real lemma" which gives necessary and sufficient conditions for a rational transfer-function to be positive-real as a matrix condition in terms of the state-space realisation. The reactance extraction technique appears to have originated in a paper by Youla and Tissi [48], which deals with the rational bounded-real scattering matrix synthesis problem.

In the research work on electrical network synthesis, special attention has been paid to the biquadratic functions [24,25,23,34,44,45,42], where the impedance is given by

$$Z(s) = \frac{a_2 s^2 + a_1 s + a_0}{d_2 s^2 + d_1 s + d_0},$$

($a_i \geq 0$ and $d_i \geq 0$). For the biquadratic impedance function to be positive real, it is necessary and sufficient to have $(\sqrt{a_2 d_0} - \sqrt{a_0 d_2})^2 \leq a_1 d_1$ [23]. Biquadratic functions have been used as an important test case for the question of minimal realisation.

In [34], Seshu proved that at least two resistors are required for a transformerless realisation of a *biquadratic minimum* function, i.e. a biquadratic function that is minimum. (This result was also given by Reza in [32].) Seshu also proved that a transformerless realisation of any minimum function requires at least three reactive elements. The author went on to prove that, for a *biquadratic minimum* function, seven elements are generally required, except for the special cases $Z(0) = 4Z(\infty)$ and $Z(\infty) = 4Z(0)$, which are realisable with a five-element bridge structure. In fact, the seven-element realisations turned out to be the modified Bott-Duffin realisations [29,40]. Following [34], it is sufficient to realise a general biquadratic function using eight elements (with one resistor to reduce a positive-real function to a minimum function). Whether it is necessary to use eight elements is still an open question.

At present, there exists no general procedure for realising biquadratic functions with the least number of elements without transformers. Given the lower order, it is very often the case that a census approach is used to cover all the possible combinations when the network structure or the number of elements is fixed (e.g. a five-element bridge network with 3 reactive elements). One attempt to generalise all biquadratic impedance functions realisable with one inductor and one capacitor (minimum reactive) without using a census approach was made by Auth [2,3]. He formulated the problem as a three-port network synthesis problem and provided certain conditions on the physical realisability of the three-port resistive network that is terminated by one inductor and one capacitor. His approach combines elements from reactance extraction and transformerless synthesis. However, it seems that there is no general method to systematically check the conditions on the physical realisability that Auth derived. Also his direct use of Tellegen's form means that six resistors are needed [41] (see Section 4.2). In Section 4, we re-consider Auth's problem and derive a more explicit result. In particular, we show that only four dissipative elements (resistors or dampers) are needed.

# 4 Transformerless Second-order Minimum Reactance Synthesis

This section considers the sub-class of biquadratic functions realisable with one spring, one inerter, an arbitrary number of dampers with no levers (transformers),

which is exactly the problem considered by Auth [2, 3] under the force-current analogy. Here, we provide a more explicit characterisation of this class.

### 4.1 Problem Formulation

We consider a mechanical one-port network consisting of an arbitrary number of dampers, one spring and one inerter. We can arrange the network in the form of Fig. 4 where $Q$ is a three-port network containing all the dampers. We bring in a mild assumption that the one-port has a well-defined admittance and the network $Q$ has a well-defined impedance. As in the proof of [36, Theorem 8.1/2] we can derive an explicit form for the impedance matrix. This is defined by

$$\begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{bmatrix} = \begin{bmatrix} X_1 & X_4 & X_5 \\ X_4 & X_2 & X_6 \\ X_5 & X_6 & X_3 \end{bmatrix} \begin{bmatrix} \hat{F}_1 \\ \hat{F}_2 \\ \hat{F}_3 \end{bmatrix} =: X \begin{bmatrix} \hat{F}_1 \\ \hat{F}_2 \\ \hat{F}_3 \end{bmatrix} \tag{1}$$

where $X$ is a non-negative definite matrix ( $\hat{}$ denotes the Laplace transform). Setting $\hat{F}_3 = -bs\hat{v}_3$ and $\hat{F}_2 = -\frac{k}{s}\hat{v}_2$, and eliminating $\hat{v}_2$ and $\hat{v}_3$ gives the following expression for the admittance

$$Y(s) = \frac{\hat{F}_1}{\hat{v}_1} = \frac{bX_3s^2 + \left[1 + kb(X_2X_3 - X_6^2)\right] s + kX_2}{b(X_1X_3 - X_5^2)s^2 + (X_1 + kb\det X)s + k(X_1X_2 - X_4^2)} \tag{2}$$

where $\det X = X_1X_2X_3 - X_1X_6^2 - X_2X_5^2 - X_3X_4^2 + 2X_4X_5X_6$. Note that $X_1 = 0$ requires that $X_4 = X_5 = 0$ for non-negative definiteness which means that the admittance does not exist. Thus the assumption of existence of the admittance requires that $X_1 > 0$.

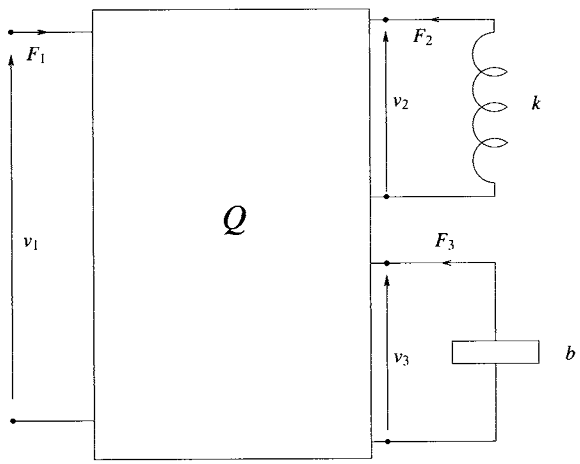


**Fig. 4.** Three-port damper network terminated with one inerter and one spring

The values of $b$ and $k$ can be set to 1 and the following scalings are carried out: $X_1 \to R_1$, $kX_2 \to R_2$, $bX_3 \to R_3$, $\sqrt{k}X_4 \to R_4$, $\sqrt{b}X_5 \to R_5$ and $\sqrt{kb}X_6 \to R_6$. The resulting admittance is

$$Y(s) = \frac{R_3 s^2 + \left[1 + (R_2 R_3 - R_6^2)\right] s + R_2}{(R_1 R_3 - R_5^2)s^2 + (R_1 + \det R)s + (R_1 R_2 - R_4^2)} \tag{3}$$

and

$$R := \begin{bmatrix} R_1 & R_4 & R_5 \\ R_4 & R_2 & R_6 \\ R_5 & R_6 & R_3 \end{bmatrix} = T \begin{bmatrix} X_1 & X_4 & X_5 \\ X_4 & X_2 & X_6 \\ X_5 & X_6 & X_3 \end{bmatrix} T,$$

where

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{k} & 0 \\ 0 & 0 & \sqrt{b} \end{bmatrix} \tag{4}$$

and $R$ is non-negative definite. From the expression $\det(R) = R_1R_2R_3 - R_1R_6^2 - R_2R_5^2 - R_3R_4^2 + 2R_4R_5R_6$, we note that (3) depends on $\text{sign}(R_4R_5R_6)$ but not on the individual signs of $R_4$, $R_5$ and $R_6$.

The reactance extraction approach to network synthesis [1,48] allows the following conclusion to be drawn: any positive-real biquadratic (immitance) function should be realisable in the form of Fig. 4 for some non-negative definite $X$. It is also known that any non-negative definite matrix $X$ can be realised as the driving-point impedance of a network consisting of dampers and levers (analogously, resistors and transformers) [10, Chapter 4, pages 173–179]. We now examine the question of the additional restrictions that are imposed when no transformers are allowed in $Q$.

### 4.2 Transformerless Realisation and Paramountcy

This section reviews the concept of paramountcy and its role in transformerless synthesis. We also state some relevant results from [13,14] which will be needed for our later results.

A matrix is defined to be *paramount* if its principal minors, of all orders, are greater than or equal to the absolute value of any minor built from the same rows [11,35]. It has been shown that paramountcy is a necessary condition for the realisability of an $n$-port resistive network without transformers [11, 35]. In general, paramountcy is not a sufficient condition for the realisability of a transformerless resistive network and a counter-example for $n = 4$ was given in [12,46]. However, in [41, pp.166–168], it was proven that paramountcy is necessary and sufficient for the realisability of a resistive network without transformers with order less than or equal to three ($n \leq 3$). The construction of [41] for the $n = 3$ case makes use of the network containing six resistors shown in Fig. 5. It is shown that this circuit is sufficient to realise any paramount matrix subject to judicious relabelling of terminals and changes of polarity. A reworking (in English) of Tellegen's proof is given in [13].

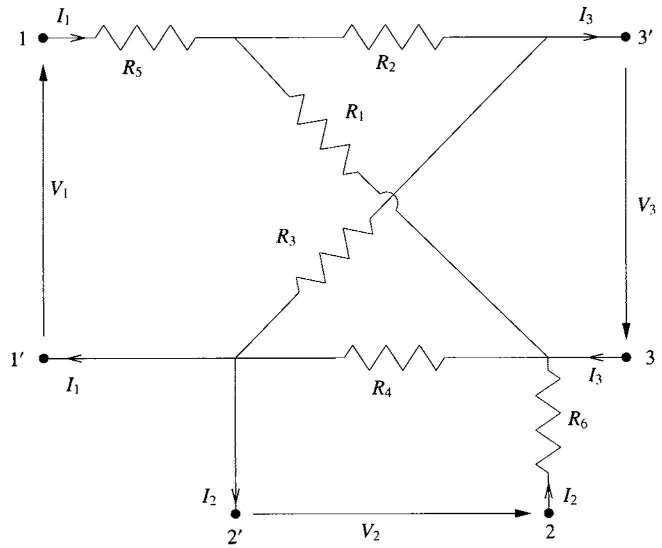


**Fig. 5.** Tellegen's circuit for the construction of resistive 3-ports without transformers

In the next two lemmas we establish a necessary and sufficient condition for a third-order non-negative definite matrix

$$R = \begin{bmatrix} R_1 & R_4 & R_5 \\ R_4 & R_2 & R_6 \\ R_5 & R_6 & R_3 \end{bmatrix} \tag{5}$$

to be reducible to a paramount matrix using a diagonal transformation. See [13, 14] for the proofs.

**Lemma 1.** *Let $R$ be non-negative definite. If any first- or second-order minor of $R$ is zero, then there exists an invertible $D = diag\{1, x, y\}$ such that $DRD$ is a paramount matrix.*

**Lemma 2.** *Let $R$ be non-negative definite and suppose that all first- and second-order minors are non-zero. Then there exists an invertible $D = diag\{1, x, y\}$ such that $DRD$ is a paramount matrix if and only if one of the following holds:*

*(i)* $R_4R_5R_6 < 0$;
*(ii)* $R_4R_5R_6 > 0$, $R_1 > \frac{R_4R_5}{R_6}$, $R_2 > \frac{R_4R_6}{R_5}$ *and* $R_3 > \frac{R_5R_6}{R_4}$;
*(iii)* $R_4R_5R_6 > 0$, $R_3 < \frac{R_5R_6}{R_4}$ *and* $R_1R_2R_3 + R_4R_5R_6 - R_1R_6^2 - R_2R_5^2 \geq 0$;
*(iv)* $R_4R_5R_6 > 0$, $R_2 < \frac{R_4R_6}{R_5}$ *and* $R_1R_2R_3 + R_4R_5R_6 - R_1R_6^2 - R_3R_4^2 \geq 0$;
*(v)* $R_4R_5R_6 > 0$, $R_1 < \frac{R_4R_5}{R_6}$ *and* $R_1R_2R_3 + R_4R_5R_6 - R_3R_4^2 - R_2R_5^2 \geq 0$.

### 4.3 Synthesis of Biquadratic Functions with Restricted Complexity

This section derives a necessary and sufficient condition for the realisability of an admittance function using one spring, one inerter, an arbitrary number

of dampers and no levers (transformers) (Theorem 1). The proof relies on the results of Section 4.2 and the construction of [41]. A stronger version of the sufficiency part of this result, which shows that at most four dampers are needed, is given in Theorem 2 with explicit circuit constructions. Singular cases are treated in Theorem 3.

**Lemma 3.** *A positive-real function $Y(s)$ can be realised as the driving-point admittance of a network in the form of Fig. 4, where $Q$ has a well-defined impedance and is realisable with dampers only and $b, k \neq 0$, if and only if $Y(s)$ can be written in the form of*

$$Y(s) = \frac{R_3 s^2 + \left[1 + (R_2 R_3 - R_6^2)\right] s + R_2}{(R_1 R_3 - R_5^2)s^2 + (R_1 + \det R)s + (R_1 R_2 - R_4^2)}, \qquad (6)$$

*where*

$$R = \begin{bmatrix} R_1 & R_4 & R_5 \\ R_4 & R_2 & R_6 \\ R_5 & R_6 & R_3 \end{bmatrix}$$

*is non-negative definite, and there exists an invertible diagonal matrix $D = diag\{1, x, y\}$ such that $DRD$ is paramount.*

*Proof:* (Only if.) As in Section 4, we can write the impedance of $Q$ in the form of (1). Since $Q$ is realised using dampers only (no transformers), we claim that the matrix $X$ in (3) is paramount. The transformation to (3), as in Section 4, now provides the required matrix $R$ with the property that $X = DRD$ is paramount where $x = 1/\sqrt{k}$ and $y = 1/\sqrt{b}$.

(If.) If we define $k = 1/x^2$ and $b = 1/y^2$, then $X = DRD$ is paramount. Using the construction of Tellegen (see Section 4.2, Fig. 5), we can find a network consisting of 6 dampers and no transformers with impedance matrix equal to $X$. Using this network in place of $Q$ in Fig. 4 provides a driving-point admittance given by (2) which is equal to (6) after the same transformation of Section 4. ■

We now combine Lemmas 1, 2 and 3 to obtain the following theorem.

**Theorem 1.** *A positive-real function $Y(s)$ can be realised as the driving-point admittance of a network in the form of Fig. 4, where $Q$ has a well-defined impedance and is realisable with dampers only and $b, k \neq 0$, if and only if $Y(s)$ can be written in the form of (6) and $R$ satisfies the conditions of either Lemma 1 or Lemma 2.*

In Theorem 2, we provide specific realisations for the $Y(s)$ in Theorem 1 for all cases where $R$ satisfies the conditions of Lemma 2. The realisations are more efficient than the construction of Tellegen (see Section 4.2, Fig. 5) in that only four dampers are needed. The singular cases satisfying the conditions of Lemma 1 are also treated in Theorem 3.

**Theorem 2.** *Let*

$$Y(s) = \frac{R_3 s^2 + \left[1 + (R_2 R_3 - R_6^2)\right] s + R_2}{(R_1 R_3 - R_5^2)s^2 + (R_1 + \det R)s + (R_1 R_2 - R_4^2)} \tag{7}$$

*where*

$$R := \begin{bmatrix} R_1 & R_4 & R_5 \\ R_4 & R_2 & R_6 \\ R_5 & R_6 & R_3 \end{bmatrix}$$

*is non-negative definite and satisfies the conditions in Lemma 2. Then $Y(s)$ can be realised with one spring, one inerter and four dampers in the form of Fig. 6(a)–6(e).*

*Proof:* Fig. 6(a)–6(e) correspond to Cases (i)–(v) in Lemma 2, respectively. Explicit formulae can be given for the constants in each circuit arrangement. Here we consider only the case of Fig. 6(a).

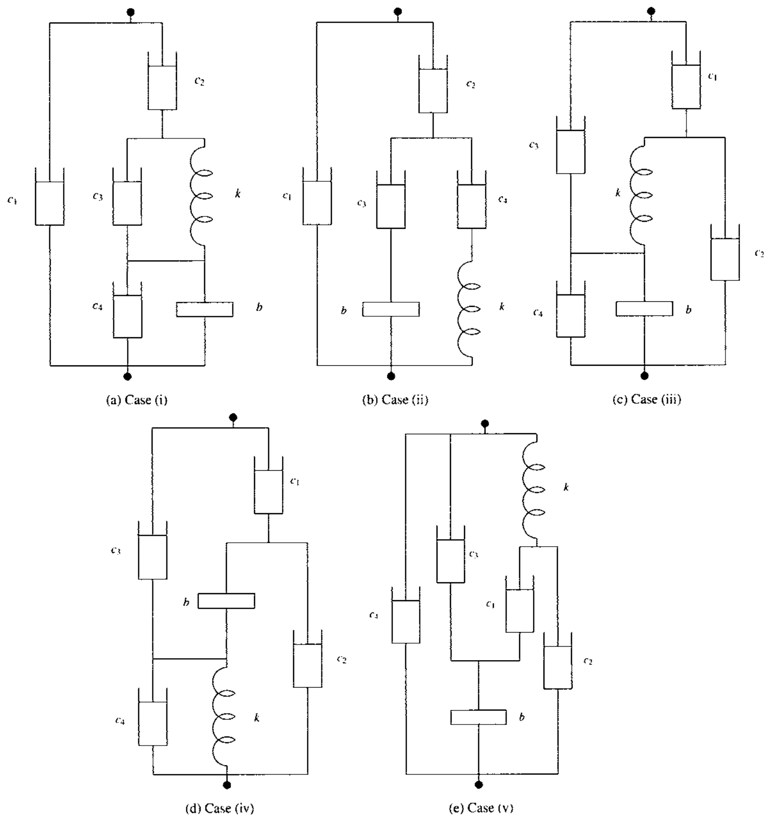


**Fig. 6.** Five circuit arrangements of Theorem 2

If $R_4R_5R_6 < 0$, $Y(s)$ can be realised in the form of Fig. 6(a) with

$$c_1 = \frac{1}{R_1 - \frac{R_4R_5}{R_6}}, \qquad c_2 = \frac{(R_3R_4 - R_5R_6)(R_4R_6 - R_2R_5)}{\det R(R_1R_6 - R_4R_5)},$$
$$c_3 = \frac{R_5^2(R_2 - \frac{R_4R_6}{R_5})}{(R_1R_6 - R_4R_5)^2}, \qquad c_4 = \frac{R_4^2(R_3 - \frac{R_5R_6}{R_4})}{(R_1R_6 - R_4R_5)^2},$$
$$b = \frac{(R_3R_4 - R_5R_6)^2}{(R_1R_6 - R_4R_5)^2}, \qquad k = \frac{(R_4R_6 - R_2R_5)^2}{(R_1R_6 - R_4R_5)^2}.$$

These formulae were derived directly in [15]. They can also be checked by direct substitution. See [15] for the procedures and the expressions of other cases. (A similar procedure has appeared in [13, 14].) ■

**Theorem 3.** *Let*

$$Y(s) = \frac{R_3s^2 + \left[1 + (R_2R_3 - R_6^2)\right]s + R_2}{(R_1R_3 - R_5^2)s^2 + (R_1 + \det R)s + (R_1R_2 - R_4^2)}$$

*where $R$ as defined in (5) is non-negative definite. If one or more of the first- or second-order minors of $R$ is zero, then $Y(s)$ can be realised with at most one spring, one inerter and three dampers.*

*Proof:* The proof is omitted for brevity. See [15] for details. ■

### 4.4 Example of Non-realisability

We now provide an explicit example of a biquadratic function which cannot be realised with two reactive elements and no transformers. First of all, we need to establish the following result.

**Theorem 4.** *The positive-real biquadratic function*

$$Y(s) = \frac{1}{h} \cdot \frac{a_0s^2 + a_1s + 1}{d_0s^2 + d_1s + 1} \tag{8}$$

*can be realised in the form of (3), equivalently Fig. 4, for a given non-negative definite $R$ if and only if $R_2$ satisfies*

$$R_2 \geq \max\left\{a_1^{-1}, d_1^{-1}, d_0/(a_0d_1)\right\}, \tag{9}$$
$$0 \leq 1 - a_1R_2 + a_0R_2^2, \tag{10}$$

*with $R_4^2$ determined by*

$$(a_1^2 - 4a_0)R_4^4 - \tag{11}$$
$$2hR_2\left((a_1d_0 - 2a_0d_1 + a_0a_1)R_2 + 2(a_0 - d_0) + a_1(d_1 - a_1)\right)R_4^2 \tag{12}$$
$$+R_2^2h^2\left((a_0 - d_0)R_2 + d_1 - a_1\right)^2 = 0 \tag{13}$$

*and satisfying*

$$hR_2(\frac{d_0}{a_0} - 1) \leq R_4^2 \leq hR_2(d_1 R_2 - 1), \tag{14}$$

*and $R_1$, $R_3$, $R_5$ and $R_6$ are determined by (15)–(18) as follows*

$$R_1 = h + \frac{R_4^2}{R_2}, \tag{15}$$

$$R_3 = a_0 R_2, \tag{16}$$

$$R_5^2 = h(a_0 - d_0)R_2 + a_0 R_4^2, \tag{17}$$

$$R_6^2 = 1 - a_1 R_2 + a_0 R_2^2, \tag{18}$$

*with $sgn(R_4 R_5 R_6) = sgn(P)$ where*

$$P := h(d_1 - a_1)R_2 + h(a_0 - d_0)R_2^2 + 2a_0 R_2 R_4^2 - a_1 R_4^2. \tag{19}$$

*Proof:* Equating (3) and (8), we have

$$h = \frac{R_1 R_2 - R_4^2}{R_2} = R_1 - \frac{R_4^2}{R_2}, \tag{20}$$

$$a_0 = \frac{R_3}{R_2}, \tag{21}$$

$$d_0 = \frac{R_1 R_3 - R_5^2}{R_1 R_2 - R_4^2}, \tag{22}$$

$$a_1 = \frac{1}{R_2} + R_3 - \frac{R_6^2}{R_2}, \tag{23}$$

$$d_1 = \frac{R_1 + \det R}{R_1 R_2 - R_4^2}. \tag{24}$$

Equations (15)–(18) then follow from (20)–(23). Substituting (15)–(18) into (24) gives

$$2R_4 R_5 R_6 = h(d_1 - a_1)R_2 + h(a_0 - d_0)R_2^2 + 2a_0 R_2 R_4^2 - a_1 R_4^2. \tag{25}$$

Thus the sign of $R_4 R_5 R_6$ is the same as the sign of $P$ defined in (19). By squaring both sides of (25), substituting from (17,18) and rearranging the terms, we obtain (11).

From (17), we can see that the non-negativity of $R_5^2$ is equivalent to the lower inequality in (14). The non-negativity of $R_6^2$ in (18) is equivalent to (10).

To ensure the non-negative definiteness of $R$, it is necessary that the principal minors are non-negative. Given a non-negative $R_2$, the non-negativity of $R_1$, $R_3$, $R_1 R_2 - R_4^2$ and $R_1 R_3 - R_5^2$ is guaranteed by (15), (16), (20) and (22). Substituting from (16) and (18), we have $R_2 R_3 - R_6^2 = a_0 R_2^2 - (1 - a_1 R_2 + a_0 R_2^2) = a_1 R_2 - 1$, which shows the necessity of the inequality

$$R_2 \geq a_1^{-1}. \tag{26}$$

Substituting (20) into (24) and rearranging the terms, we have $R_2 \det R = h(d_1R_2 - 1)R_2 - R_4^2$, and therefore

$$\det R \geq 0 \Leftrightarrow h(d_1R_2 - 1)R_2 \geq R_4^2 \tag{27}$$

which shows the necessity of the upper inequality in (14). For (27) to have a solution it is necessary that

$$R_2 \geq d_1^{-1}. \tag{28}$$

For the range defined in (14) to be non-empty, it is necessary that

$$R_2 \geq d_0/(a_0d_1). \tag{29}$$

Combining (26), (28) and (29) gives (9). ■

Now we will show in the example below that it is not always possible to realise a biquadratic in the form of Fig. 4 without transformers (levers).

**Example** (non-realisability). Consider the admittance function

$$Y(s) = \frac{2s^2 + s + 1}{s^2 + s + 1},$$

which takes the form (8) with $a_0 = 2$, $a_1 = d_0 = d_1 = h = 1$. Since $a_1d_1 > (\sqrt{a_0} - \sqrt{d_0})^2$, $Y(s)$ is positive-real. Now we apply the procedure in Theorem 4. By (9) and (14), it is necessary to have $R_2 \geq 1$ and

$$0 \leq R_4^2 \leq R_2(R_2 - 1). \tag{30}$$

Note that (10) is redundant in this case. For a particular $R_2$, $R_4^2$ is solved by (11). Then $R_1$, $R_3$, $R_5$ and $R_6$ are determined by (15)–(18). The solution of $R_4^2$ from (11) and the upper bound in (14) are plotted in Fig. 7. Therefore, we can see that any $R_2$ sufficiently large (in fact $R_2 \geq 1.5$) gives a non-negative definite $R$ satisfying Theorem 4.

Now we would like to show that it is not possible to realise this admittance function in the form of Fig. 4 without transformers (levers). First, we note from (19) that $P = R_2^2 + (4R_2 - 1)R_4^2 \geq 0$ for all $R_2 \geq 0$. Therefore $R_4R_5R_6 > 0$ for any admissible $R_2$. By substituting from (17) and (18), it is easy to show that

$$\frac{R_5^2R_6^2}{R_4^2} - R_3^2 = \frac{1}{R_4^2}\left(R_2(R_2+1) + 2R_4^2 + 2R_2\left(R_2(R_2-1) - R_4^2\right)\right) \geq 0,$$

which implies that $R_3 < R_5R_6/R_4$ for any admissible $R_2$ . However,

$$\begin{aligned} R_1R_2R_3 &+ R_4R_5R_6 - R_1R_6^2 - R_2R_5^2 \\ &= \frac{1}{2R_2}\left((R_2-2)R_4^2 - R_2(R_2^2 - 2R_2 + 2)\right) \\ &\leq \frac{1}{2R_2}\left((R_2-2)R_2(R_2-1) - R_2(R_2^2 - 2R_2 + 2)\right) \\ &= -\frac{R_2}{2} < 0, \end{aligned}$$

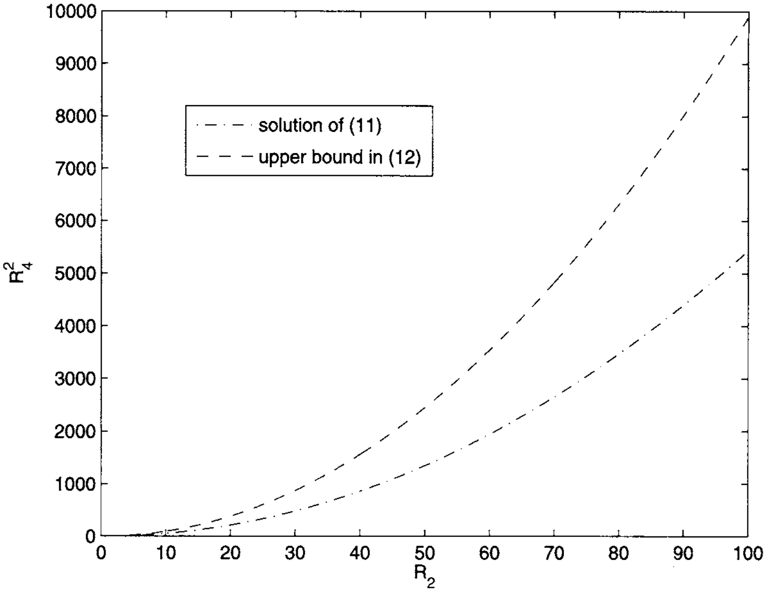


**Fig. 7.** Solution of $R_4^2$ and upper bound versus $R_2$

where the first inequality made use of (30). Therefore the second condition of Case (iii) in Lemma 2 fails for any admissible $R_2$. Therefore, $Y(s)$ cannot be realised in the form of Fig. 4 without transformers (levers).

## 5 Conclusions

The theme of this paper is the application of electrical circuit synthesis to mechanical networks. Relevant results from the field of passive networks have been surveyed. It was pointed out that the problem of minimal realisation (in terms of the number of elements used) is still unsolved, and that this is an important question for mechanical implementation. The class of biquadratic positive-real functions was highlighted as an interesting test case. For this class, an explicit procedure was provided to determine if a given function can be realised with two reactive elements and no transformers.

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