Realizability of $n$-Port Resistive Networks with $2n$ Terminals

The 9th Asian Control Conference (ASCC 2013)

Michael Z. Q. Chen

Department of Mechanical Engineering
The University of Hong Kong

June 24, 2013
Outline

1. Introduction
2. Problem Formulation
3. Main Results
4. Conclusion
Outline

1. Introduction
2. Problem Formulation
3. Main Results
4. Conclusion
One-Port Passive Networks

Definition

A one-port network is passive if the behaviors of its two terminals satisfies:

\[ \int_{-\infty}^{T} i(t)v(t)dt \geq 0, \]

for all physically acceptable current \( i(t) \), voltage \( v(t) \), and all \( T \).

Definition

$Z(s)$ is defined to be a positive-real function if $Z(s)$ is analytic and $\Re Z(s) \geq 0$ in $\Re [s] > 0$. 

The admittance (or impedance) of a passive network must be positive-real, i.e. $\Re(Y(j\omega)) \geq 0$ for all $\omega$. (Not too difficult to prove.)
Defining admittance as $Y(s) = I(s)/V(s)$, and impedance as $Z(s) = V(s)/I(s)$, we can obtain the following conclusion.

**Theorem**

The immitance (impedance or admittance) of any passive network must be a positive-real function.

What about the converse?

In 1931, O. Brune established a procedure that can realize any positive-real function with passive resistors, inductors, capacitors, and transformers.

A one-port network is passive if and only if its immittance is a positive-real function. Moreover, any positive-real immittance can be realized with finite number of resistors, inductors, capacitors, and transformers.
In 1949, R. Bott and R. J. Duffin established a new realization procedure for any positive-real function without transformers.

Theorem

Any positive-real immittance can be realized by a transformerless network containing finite number of resistors, capacitors, and inductors.

The resistor, capacitor, and inductor are called as three basic electrical elements for passive networks.
Force-Current Analogy:

- current ↔ force
- voltage ↔ velocity
- electrical ground ↔ mechanical ground

Passivity:

$$\int_{-\infty}^{T} F(t)v(t)dt \geq 0$$

Mechanical Immittances:

$$Z(s) = \hat{v}/\hat{F}, \quad Y(s) = \hat{F}/\hat{v}$$
Replacethespringanddamperwithageneralpositive-realimpedance $Z(s)$.

But is $Z(s)$ physically realizable?

$Z(s)$ must be positive-real.

But is $Z(s)$ always physically realizable?
Analogy of Basic Elements (Force-Current Method)

\[ v = Ri \quad \text{resistor} \leftrightarrow \text{damper} \quad cv = F \]
\[ v = L \frac{di}{dt} \quad \text{inductor} \leftrightarrow \text{spring} \quad kv = \frac{dF}{dt} \]
\[ C \frac{dv}{dt} = i \quad \text{capacitor} \leftrightarrow \text{mass} \quad m \frac{dv}{dt} = F \]

The Exceptional Nature of the Mass Element

Newton's Second Law gives the following network interpretation of the mass element:

- One terminal is the centre of mass,
- Other terminal is a fixed point in the inertial frame.

Hence, the mass element is analogous to a grounded capacitor.

Standard network symbol for the mass element:

\[ v_1 = 0 \quad v_2 = F \]
The (ideal) inerter is a two-terminal mechanical device with the property that the equal and opposite force applied at the terminals is proportional to the relative acceleration between them, that is, \( F = b \ddot{v} \) where \( \dot{v} = \dot{v}_1 - \dot{v}_2 \).

Applications of the Inerter

1. Vehicle suspension systems control
2. Motorcycle steering control
3. Building systems control
4. Train suspension systems control
5. .......
The Missing Mechanical Circuit Element

Michael Z.Q. Chen, Christos Papageorgiou, Frank Scheibe, Fu-Cheng Wang, and Malcolm C. Smith

Abstract

In 2008, two articles in Autosport revealed details of a new mechanical suspension component with the name “J-damper” which had entered Formula One Racing and which was delivering significant performance gains in handling and grip. From its first mention in the 2007 Formula One “spy scandal” there was much speculation about what the J-damper actually was. The Autosport articles revealed that the J-damper was in fact an “inerter” and that its origin lay in academic work on mechanical and electrical circuits at Cambridge University. This article aims to provide an overview of the background and origin of the inerter, its application, and its intimate connection with the classical theory of network synthesis.

The theory of passive electrical network synthesis can be completely transplanted into mechanical networks.

Mechanical Networks Synthesis and Existing Problems

The interest in the investigation is renewed

Theory of passive network synthesis

Passive mechanical systems design

Redundant elements limit its practical use
The \( n \)-port network is passive if the behaviors of its terminals satisfies:

\[
\int_{-\infty}^{T} I(t)^T V(t) dt \geq 0,
\]

for all physically acceptable currents \( I(t) = [i_1(t), i_2(t), \ldots, i_n(t)]^T \), voltages \( V(t) = [v_1(t), v_2(t), \ldots, v_n(t)]^T \), and all \( T \).

Definition

A real-rational matrix $B(s)$ is defined to be positive real if all elements of $B(s)$ are analytic in $\text{Re}[s] > 0$ and $B^T(s) + B(s) \geq 0$ for $\text{Re}[s] > 0$.

Theorem

*The n-port network is passive if and only if its immittance matrix is positive real. Moreover, any positive real immittance matrix is realizable by an n-port network containing a finite number of resistors, capacitors, inductors, and transformers.*

Unlike one-port networks, the transformerless realization problem of passive multi-port networks is far from being solved even for the multi-port resistive networks.

The investigation on $n$-port resistive networks can provide guidance on the discussion of more general transformerless $n$-port networks. So it has been one of the classical topics in the theory of passive network synthesis.
Outline

1. Introduction
2. Problem Formulation
3. Main Results
4. Conclusion
Realizations of the $n \times n$ Real Symmetric Matrix

Consider an $n \times n$ real symmetric matrix in the form of

$$\begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} & \cdots & Y_{1,n-1} & Y_{1n} \\
Y_{12} & Y_{22} & Y_{23} & \cdots & Y_{2,n-1} & Y_{2n} \\
Y_{13} & Y_{23} & Y_{33} & \cdots & Y_{3,n-1} & Y_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
Y_{1,n-1} & Y_{2,n-1} & Y_{3,n-1} & \cdots & Y_{n-1,n-1} & Y_{n-1,n} \\
Y_{1n} & Y_{2n} & Y_{3n} & \cdots & Y_{n-1,n} & Y_{nn}
\end{bmatrix}. \quad (1)$$

**Definition**

A real symmetric matrix, with each principal minor not less than the absolute value of any non-principle minor formed by the same rows, is called a *paramount matrix*.

**Theorem**

*If $Y$ in the form of (1) is the admittance of an $n$-port resistive network, then $Y$ must be paramount.*
Realizations of the $n \times n$ Real Symmetric Matrix

**Assumption**

*Since for resistive networks internal vertices (the vertices which are not terminals) can be eliminated by the generalized star-mesh transformation, we assume that the resistive network always contains vertices that are all terminals. Therefore, the number of terminals ranges from $n + 1$ to $2n.*

- The realization problem for the admittance matrix to be realizable by $n$-port resistive networks containing $n+1$ terminals has been successfully solved [1–2].

- The problem for networks containing $n + p$ terminals where $p > 1$ terminals needs to be further investigated [3].


The realizability condition of resistive networks containing $2n$ distinct terminals is independent of the structures and orientations of the ports, which appears to be simpler.

To solve the realization problem of resistive networks containing $n + p$ terminals with $1 < p < n - 1$, it is only necessary to investigate the networks that cannot be equivalent to $2n$-terminal networks.

**Problem**

Consider an $n \times n$ real symmetric matrix in the form of (1). What is the necessary and sufficient condition for $Y(s)$ to be realizable by an $n$-port resistive network containing $2n$ distinct terminals? What about the case when $n = 3$?
Outline

1. Introduction
2. Problem Formulation
3. Main Results
4. Conclusion
Main Result 1:

For a real symmetric $n \times n$ matrix $Y$, it is realizable as the admittance of an $n$-port resistive network $N$ with $2n$ terminals if and only if there exists a $(2n - 1) \times (n - 1)$ matrix $P$ such that a set of inequalities in the form of

$$N_{ij} \leq y_{ij} \leq M_{ij}$$

with $i \geq j$ is satisfied, where $M_{ij}$ and $N_{ij}$ are in terms of the row vectors of $P$, and $M_{ii} = +\infty$. 
A Necessary and Sufficient Condition for Realization

Theorem 1

A real symmetric \( n \times n \) matrix \( Y \) in the form of (1) can be realized as an \( n \)-port resistive network \( N \) with \( 2n \) terminals if and only if there exists a real \((2n-1) \times (n-1)\) matrix \( P \) in the form of

\[
P = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} \end{bmatrix}
\]

with

\[
\gamma_k := \begin{bmatrix} p_1^{(k)} & p_{n+1}^{(k)} & \cdots & p_k^{(k)} & p_{n+k}^{(k)} & p_{k+1}^{(k)} & 0 & \cdots & p_{n-1}^{(k)} & 0 & p_n^{(k)} \end{bmatrix}^T
\]

for \( 1 \leq k \leq n - 1 \), such that

\[
\max\{ - (\alpha_i - \beta_{i-1})^T (\alpha_j - \beta_j), - (\alpha_i - \beta_i)^T (\alpha_j - \beta_{j-1}) \} \leq y_{ij} \\
\leq \min\{ - (\alpha_i - \beta_{i-1})^T (\alpha_j - \beta_{j-1}), - (\alpha_i - \beta_i)^T (\alpha_j - \beta_j) \}
\]

for \( 1 \leq i < j \leq n \) and

\[
y_{ii} \geq - (\alpha_i - \beta_{i-1})^T (\alpha_i - \beta_i), \quad 1 \leq i = j \leq n,
\]

where \( \alpha_i^T \) is the \((2l - 1)\)th row of \( P \) with \( 1 \leq l \leq n \), \( \beta_m^T \) is the \( 2m \)-th row of \( P \) with \( 1 \leq m \leq n - 1 \), and \( \beta_0 = \beta_n := 0_{(n-1) \times 1} \).
Main Result 2:

If a real symmetric $n \times n$ matrix $Y$ satisfies the condition of Main Result 1, then the conductances of the resistors $g_{h,j}$ can be expressed in terms of $y_{ij}$ and the row vectors of the matrix $P$, where $g_{h,j}$ denotes the conductance of the resistor connecting Terminals $A_h$ and $A_j$, and the orientations of the ports are from $A_{2k-1}$ to $A_{2k}$ with $1 \leq k \leq n$. 

![Diagram of an n-port resistive network with 2n terminals](image)
The Values of the Elements

**Theorem 2**

If a real symmetric $n \times n$ matrix $Y$ in the form of (1) can be realized as an $n$-port resistive network with $2n$ terminals, where the orientations of the ports are from vertex $A_{2k-1}$ to $A_{2k}$ with $1 \leq k \leq n$, that is, $Y$ satisfies the condition of Theorem 1, then the conductances of the resistors of the edges are given by

\[
g_{2r-1,2s} = y_{r,s} + (\alpha_r - \beta_{r-1})^T (\alpha_s - \beta_s), \quad 1 \leq r \leq s \leq n,
\]
\[
g_{2r-1,2s-1} = -y_{r,s} - (\alpha_r - \beta_{r-1})^T (\alpha_s - \beta_{s-1}), \quad 1 \leq r < s \leq n,
\]
\[
g_{2r,2s-1} = y_{r,s} + (\alpha_r - \beta_r)^T (\alpha_s - \beta_{s-1}), \quad 1 \leq r < s \leq n,
\]
\[
g_{2r,2s} = -y_{r,s} - (\alpha_r - \beta_r)^T (\alpha_s - \beta_s), \quad 1 \leq r < s \leq n,
\]

where $\alpha_k$ and $\beta_k$ are obtained from the parameter matrix $P$ as defined in Theorem 1, and $g_{h,l}$ denotes the conductance of the element connecting vertices $A_h$ and $A_l$ for $1 \leq h < l \leq 2n$. 
The condition of **Main Result 1** is based on the existence of a matrix $P$.

Equivalent conditions that are only in terms of the entries of the $n \times n$ matrix $Y$ needs further to be investigated.

The discussion can begin with low-order cases like $n = 3$.

The investigation on three-port resistive networks is a classical topic, and can provide important guidance on more general networks.
Tellegen’s Three-Port Resistive Networks

Any $3 \times 3$ paramount $Y$ is realizable by the following network.

The realization of three-port resistive networks with six terminals and at most five elements needs to be investigated.
Main Result 3:

A third-order real symmetric matrix $Y = (y_{ij})$ can be realized as the admittance of a three-port resistive network with six terminals and at most five positive elements if and only if a set of certain restrictions that is only in terms of $y_{ij}$ holds.

$$Y = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$
Realizability Condition

Definition

A row of a matrix $Y$ is marginally dominant if the elements of the row satisfy

$$y_{ii} = \sum_{j=1, j\neq i}^{n} |y_{ij}|.$$

Theorem 3

A third-order real symmetric matrix $Y$ can be realized as the admittance matrix of a three-port resistive network with six terminals and at most five positive elements, which is not terminal well-connected, if and only if $Y$ is dominant and one of the following three conditions holds:

1. at least two of $y_{12}$, $y_{13}$, and $y_{23}$ are zero;
2. one of $y_{12}$, $y_{13}$, and $y_{23}$ is zero, and at least two of the three rows are marginally dominant;
3. one of $y_{12}$, $y_{13}$, and $y_{23}$ is zero, denoted by $y_{ij} = 0$, and only one of the three rows is marginally dominant, which is either the $i$th row or the $j$th row.
1. The new background of passive network synthesis has been introduced.

2. A necessary and sufficient condition for any $n \times n$ admittance matrix to be realizable by an $n$-port resistive network containing $2n$ terminals has been established, which is based on the existence of a parameter matrix.

3. The values of the elements of the $n$-port resistive network have been parameterized.

4. A necessary and sufficient condition for any real symmetric matrix to be realizable as the admittance of a three-port six-terminal resistive network, which is not terminal well-connected and contains at most five positive elements, has been derived.
Thank you very much for your attention!