International Mathematical Olympiad

Preliminary Selection Contest 2012 — Hong Kong

Outline of Solutions

Answers:

1. 170	**2.** 104	**3.** $\frac{7}{13}$	**4.** 6
5. 192	**6.** 1998	**7.** 11	**8.** $-6+\sqrt[8]{6}$
9. 55440	**10.** $\frac{49}{58}$	**11.** 18	**12.** $\frac{11-\sqrt{37}}{3}$
13. 3	**14.** 673685	**15.** 6	**16.** $\frac{\sqrt{11}}{2}$
17. $6+\sqrt{35}$	**18.** 82944	**19.** 8221	**20.** 8

Solutions:

1. Since n is a two-digit number, we have $n^2 \le 99^2 = 9801 < 9999$ and so the sum of digits of n^2 is less than $9\times 4 = 36$. Since the sum of digits of n^2 is equal to the square of the sum of digits of n, the sum of digits of n is less than $\sqrt{36} = 6$. It remains to search through all the two-digit numbers whose sum of digits not greater than 5. There are 15 such numbers, namely, 10, 11, 12, 13, 14, 20, 21, 22, 23, 30, 31, 32, 40, 41, 50. By checking these numbers one by one, we know that the answer is $10+11+12+13+20+21+22+30+31=170$.

2. Note that $n^2-1=(n-1)(n+1)$. We thus want one of $n-1$ and $n+1$ to be a prime and the other to be a product of two primes. All of these primes have to be odd since $n-1$ and $n+1$ have the same parity and there exists only one even prime.

   Call a positive integer a 'semiprime' if it is the product of two distinct odd primes. The first five 'semiprimes' are thus 15, 21, 33, 35, 39. We need to look for small primes which differ from a 'semiprime' by 2. We thus, get (13, 15), (15, 17), (19, 21), (21, 23) and (31, 33) as the five smallest sets of ($n-1$, $n+1$). Therefore, the five smallest 'good' integers are 14, 16, 20, 22 and 32. Summing them up, we get the answer 104.

3. The probability that the monitor can choose his favourite seat depends on his position in the queue. If he is first in the queue, he must be able to get his favourite seat; if he is the second in the queue, there is a probability of $\frac{12}{13}$ that his favourite seat has not been chosen by the first person in the queue. Similarly, if he is k-th in the queue, the probability that he gets his favourite seat is $\frac{14-k}{13}$. Since the monitor is equally likely to occupy each position in the queue, the answer is $\frac{1}{13}\left(1+\frac{12}{13}+\frac{11}{13}+\cdots+\frac{1}{13}\right)=\frac{7}{13}$.

4. Since a positive integer leaves the same remainder as its sum of digits when divided by 9, $D(x)$ is simply the remainder when x is divided by 9. In other words, the answer is simply the remainder when F_{2012} (where F_n is the n-th Fibonacci number) is divided by 9.

   We compute the 'modulo 9 Fibonacci sequence' (i.e. the remainders when terms of the Fibonacci sequence are divided by 9): we get 1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 9, after which the next two terms are 1, 1 and hence the sequence will repeat every 24 terms. Since $2012 \equiv 20$ (mod 24), we have $F_{2012} \equiv F_{20} \equiv 6$ (mod 9).

5. Let b_i and g_i (where $1 \le i \le 5$) be the number of boys and girls in the i-th class respectively. Since there are $600 \div 5 = 120$ students in each class and at least 33 boys and girls, each b_i and g_i is between 33 and 87 inclusive. Consider the matrix:

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ g_1 & g_2 & g_3 & g_4 & g_5 \end{pmatrix}$$

   The sum of each row is 300, while the sum of each column is 120. If we circle the smaller number in each column, the sum of the circled numbers is the number of teams which can be formed.

   By the pigeonhole principle, at least three numbers are circled in the same row. Without loss of generality assume b_1, b_2 and b_3 are circled. Note that $b_1+b_2+b_3 \ge 300-2\times 87=126$. On the other hand, each circled number in the fourth and fifth column is at least 33. The sum of the circled numbers is thus at least $126+33+33=192$. Equality is possible as the matrix

$$\begin{pmatrix} 42 & 42 & 42 & 87 & 87 \\ 78 & 78 & 78 & 33 & 33 \end{pmatrix}$$

   provides one such example. The answer is thus 192.

6. Note that $8 = x^3 - 3\sqrt{2}x^2 + 6x - 2\sqrt{2} = \left(x - \sqrt{2}\right)^3$. It follows that $x = 2 + \sqrt{2}$, or $(x-2)^2 = 2$, or $x^2 - 4x + 2 = 0$. Thus $x^5 - 41x^2 + 2012 = (x^2 - 4x + 2)(x^3 + 4x^2 + 14x + 7) + 1998 = 1998$.

7. Call a positive integer 'good' if it can be expressed in the form $\frac{2^a - 2^b}{2^c - 2^d}$ where a, b, c, d are non-negative integers. Clearly, if n is 'good', then so is $2n$ because we can simply increase a and b by 1 to double the value of $\frac{2^a - 2^b}{2^c - 2^d}$.

   Note that 1, 3, 5, 7, 9 are 'good' since $1 = \frac{2^2 - 2^1}{2^2 - 2^1}$, $3 = \frac{2^3 - 2^1}{2^2 - 2^1}$, $5 = \frac{2^4 - 2^1}{2^2 - 2^0}$, $7 = \frac{2^4 - 2^1}{2^2 - 2^1}$ and $9 = \frac{2^6 - 2^0}{2^3 - 2^0}$. Hence 2, 4, 6, 8, 10 are also 'good' by the remark in the previous paragraph.

   Finally, assume $11 = \frac{2^a - 2^b}{2^c - 2^d} = \frac{2^k(2^m - 1)}{2^n - 1}$ where $m = a - b$, $n = c - d$ and $k = b - d$, with m, n positive. It follows that $11(2^n - 1) = 2^k(2^m - 1)$. Since the left hand side is odd, we have $k = 0$. Clearly, neither m nor n can be equal to 1. Thus $2^m - 1 \equiv 2^n - 1 \equiv 3$ (mod 4). As $11 \equiv 3$ (mod 4), we get a contradiction as the left hand side is congruent to 1 but the right hand side is congruent to 3 modulo 4. Thus 11 is not 'good' and so the answer is 11.

8. By completing square we have $f(x) = (x+6)^2 - 6$ and so

$$f(f(x)) = (((x+6)^2 - 6) + 6)^2 - 6 = (x+6)^4 - 6.$$

   Similarly, $f(f(f(f(x)))) = (x+6)^8 - 6$. The equation thus becomes $(x+6)^8 = 6$, with solutions $x = -6 \pm \sqrt[8]{6}$. Hence the answer is $-6 + \sqrt[8]{6}$.

9. Ignoring the rule that no two adjacent letters be the same, the answer would be $\frac{9!}{2!2!}$ as there are 9 letters including two I's and two L's. From this we must count the number of permutations with two adjacent I's or two adjacent L's.

   If the two I's are adjacent, we can treat them as one single letter and hence the number of permutations would be $\frac{8!}{2!}$. The same is true for permutations with two adjacent L's. There are overlappings between these two types of permutations though, as there are 7! permutations in which the two I's and the two L's are both adjacent. Hence the answer is

$$\frac{9!}{2! \times 2!} - \left(\frac{8}{2!} + \frac{8}{2!} - 7!\right) = 55440.$$

10. From the given equations we have $4\sin^2 x = 36\sin^2 y$ and $4\cos^2 x = \cos^2 y$. It follows that

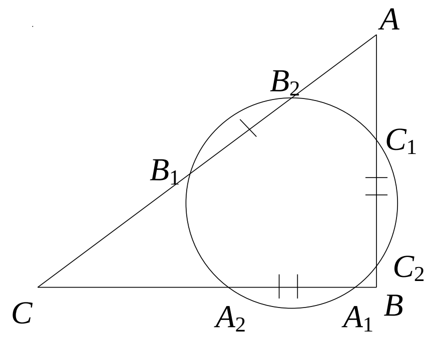
$$4 = 4\sin^2 x + 4\cos^2 x = 36\sin^2 y + \cos^2 y = 36\sin^2 y + (1-\sin^2 y)$$

and so $\sin^2 y = \frac{3}{35}$. In the same way we get $\sin^2 x = \frac{27}{35}$, $\cos^2 x = \frac{8}{35}$ and $\cos^2 y = \frac{32}{35}$. Hence

$$\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y} = \frac{2\sin x\cos x}{2\sin y\cos y} + \frac{\cos^2 x - \sin^2 x}{\cos^2 y - \sin^2 y} = 2(3)\left(\frac{1}{2}\right) + \frac{\frac{8}{35} - \frac{27}{35}}{\frac{32}{35} - \frac{3}{35}} = \frac{49}{58}.$$

11. Let d be the last digit of x, and write $x = 10c + d$. After moving the last digit to the front, the number becomes $10^{n-1}d + c$. According to the question, we have $10^{n-1}d + c = 2(10c + d)$, or $(10^{n-1} - 2)d = 19c$. Computing 10^k modulo 19 for k = 1, 2, 3, …, we get 10, 5, 12, 6, 3, 11, 15, 17, 18, 9, 14, 7, 13, 16, 8, 4, 2. Hence the smallest possible value of $n-1$ is 17. It remains to show that there is a 18-digit number with the given property. Since c is to be a 17-digit number, from $(10^{n-1} - 2)d = 19c$, we see that we should set $d = 2$ when $n = 18$. This corresponds to $c = 10526315789473684$ (hence $x = 105263157894736842$). It follows that the answer is 18.

12. Since $A_1A_2 = B_1B_2 = C_1C_2$, the circle has the same centre as the inscribed circle of ΔABC. If we let A_0 be the mid-point of A_1A_2, then A_0 is the point where the inscribed circle of ΔABC touches BC, and the same as true for B_0 and C_0. Since $AB_0 = AC_0$, $BC_0 = BA_0$ and $CA_0 = CB_0$, it is easy to find that $AB_0 = AC_0 = 2$, $BC_0 = BA_0 = 1$ and $CA_0 = CB_0 = 3$. Thus we have $AB_2 = AC_1 = 2 - \frac{x}{2}$, $BC_2 = BA_1 = 1 - \frac{x}{2}$ and $CA_2 = CB_1 = 3 - \frac{x}{2}$.

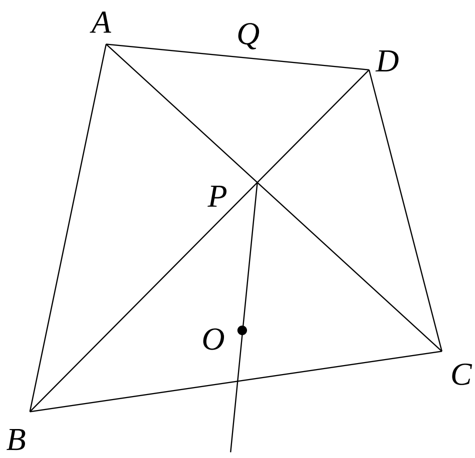


The area of ΔBC_2A_1 is thus $\frac{1}{2}\left(1-\frac{x}{2}\right)^2$. As $\sin A = \frac{4}{5}$, the area of ΔAB_2C_1 is $\frac{1}{2}\left(2-\frac{x}{2}\right)^2\left(\frac{4}{5}\right)$. Similarly, ΔCA_2B_1 has area $\frac{1}{2}\left(3-\frac{x}{2}\right)^2\left(\frac{3}{5}\right)$. Finally, since the area of ΔABC is 6, we have

$$\frac{1}{2}\left(1-\frac{x}{2}\right)^2 + \frac{1}{2}\left(2-\frac{x}{2}\right)^2\left(\frac{4}{5}\right) + \frac{1}{2}\left(3-\frac{x}{2}\right)^2\left(\frac{3}{5}\right) = 2$$

and hence $x = \frac{11 \pm \sqrt{37}}{3}$. Obviously the negative square root should be taken as $x < 3$.

13. Let OP meet AD at Q. Note that we have $OB = OC = OP$ and hence $\angle OPB + \angle PCB = 90°$. Since $\angle OPB = \angle QPD$ and $\angle PCB = \angle PDQ$ (as $PA \cdot PC = PB \cdot PD = 6$, which implies $ABCD$ is concyclic), we have $\angle QPD + \angle PDQ = 90°$.



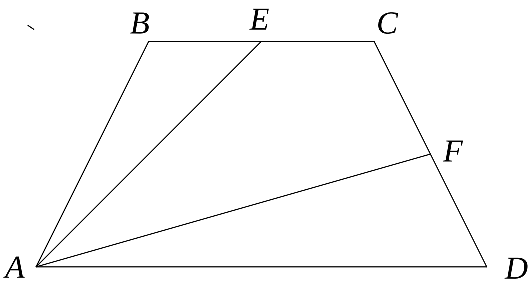
This implies OQ is perpendicular to AD. Yet it is also given that OA is perpendicular to AD. Hence O must lie on the straight line APC. As O is the circumcentre of ΔPBC, it follows that PC is a diameter of its circumcircle. Since $PC = 6$, the circumradius is 3.

14. Let x be the common difference of the arithmetic sequence a, b, c, d. We have $d = a + 3x$. The question is thus equivalent to counting the number of positive integer solutions to the equation $(a + 3x) + t = 2013$.

When $x = 1$, the equation becomes $a + t = 2010$ and there are 2009 solutions (corresponding to a = 1, 2, …, 2009). When $x = 2$, the equation becomes $a + t = 2007$ and there are 2006 solutions. Likewise, when x = 3, 4, 5, …, 669, 670, there are 2003, 2000, 1997, …, 5, 2 solutions respectively. It follows that the answer is

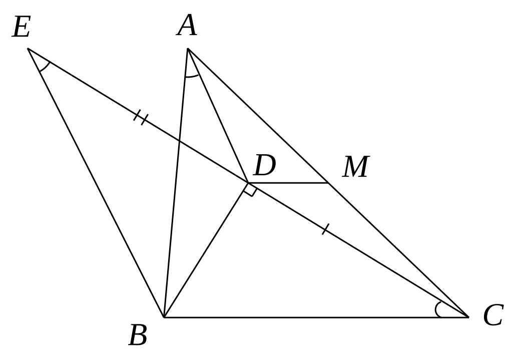
$$2009 + 2006 + \cdots + 2 = \frac{(2009 + 2)(670)}{2} = 673685.$$

15. Let the areas of the four triangles be n, $n+1$, $n+2$ and $n+3$, where n is a positive integer. The area of the quadrilateral $ABCD$ is thus $4n+6$. Note that the area of ΔBCD is four times that of ΔECF, which is at least $4n$. Hence the area of ΔABD is at most 6.

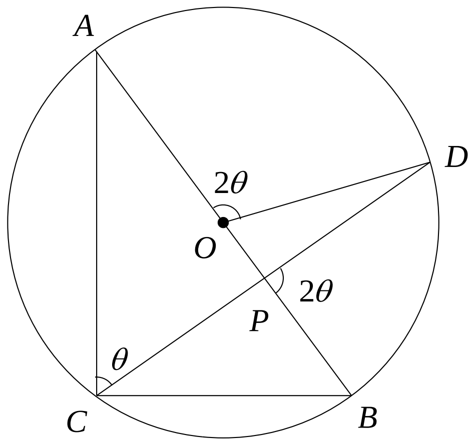


Equality can be attained when $ABCD$ is an isosceles trapezium with parallel sides $AD = 6$ and $BC = 4$, and height 2. (We can check in this case that the areas of ΔCEF, ΔABE, ΔADF and ΔAEF are 1, 2, 3 and 4 respectively, and ΔABD has area 6). The answer is thus 6.

16. Extend CD to E so that $DE = DC$. Then ΔBDE and ΔBDC are congruent so that we have $BE = BC = 6$ and $\angle BED = \angle BCD = \angle BAD$. It follows that A, D, B, E are concyclic and so $\angle EAB = \angle EDB = 90°$. Thus $AE = \sqrt{6^2 - 5^2} = \sqrt{11}$ and hence the mid-point theorem asserts that $DM = \frac{1}{2}AE = \frac{\sqrt{11}}{2}$.



17. Let O be the mid-point of AB, which is also the circumcentre of ΔABC. Extend CP to meet the circumcircle at D. Note that P is between O and B (if P is between O and A then $\angle ACP$ is less than 45° while $\angle APC$ is obtuse, contradicting $\angle APC = 2\angle ACP$).



Let $\angle ACP = \theta$. Then $\angle DPB = \angle APC = 2\angle ACP = 2\theta$ and $\angle AOD = 2\theta$. It follows that $DO = DP$, both being 3.5 (radius of the circle). Using the power chord theorem $PA \times PB = PC \times PD$, we have $PA \times (7 - PA) = 1 \times 3.5$. Solving, we get $PA = \frac{7 \pm \sqrt{35}}{2}$. The positive square root is taken as P is between O and B. It follows that $PB = \frac{7 - \sqrt{35}}{2}$ and so the answer is $\frac{7+\sqrt{35}}{7-\sqrt{35}} = 6 + \sqrt{35}$.

18. Note that, in order for the sum of any three adjacent integers after the rearrangement to be divisible by 3, any three adjacent integers must be pairwise different modulo 3. For the original positions of 1, 2, 3, there are $3! = 6$ possibilities to arrange three numbers taken modulo 3 (i.e. exactly one of these positions is to be occupied by a number divisible by 3, one by a number congruent to 1 modulo 3, etc.). The remaining numbers, taken modulo 3, are then fixed. (For example, if the three numbers occupying the original positions of 1, 2, 3 are 8, 9, 10, then the twelve numbers modulo 3 in clockwise order starting from the original position of 1 must be 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1.) For each such ‘modulo 3’ arrangement, there are $4! = 24$ ways to arrange each of the 4 numbers congruent to 0, 1 and 2 modulo 3. Hence the answer is $6 \times 24 \times 24 \times 24 = 82944$.

19. As 100140001 is of the form $10^8+14\times10^4+1$, we naturally try to factorise x^8+14x^4+1. Indeed, we have

$$\begin{aligned}x^8+14x^4+1&=x^8+2x^4+1+12x^4\\&=(x^4+1)^2+12x^4\\&=[(x^4+1)^2+4x^2(x^4+1)+4x^4]+8x^4-4x^2(x^4+1)\\&=(x^4+2x^2+1)^2-4x^2(x^4-2x^2+1)\\&=(x^4+2x^2+1)^2-(2x^3-2x)^2\\&=(x^4+2x^3+2x^2-2x+1)(x^4-2x^3+2x^2+2x+1)\end{aligned}.$$

By putting $x=10$, we get $100140001=12181\times8221$. Hence the answer is 8221.

20. Let a, b, c (where $a\le b\le c$) be the lengths of the sides of such a triangle. By Heron's formula, we have

$$\sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{c+a-b}{2}\right)\left(\frac{b+c-a}{2}\right)}=2(a+b+c),$$

which simplifies to

$$(a+b-c)(c+a-b)(b+c-a)=64(a+b+c).$$

Observe that $a+b-c$, $c+a-b$, $b+c-a$, $a+b+c$ have the same parity and hence must be even. Set $a+b-c=2r$, $c+a-b=2s$, $b+c-a=2t$, where r, s, t are positive integers with $3\le r\le s\le t$. The above equation is then reduced to

$$rst=16(r+s+t).$$

As $t<r+s+t\le3t$, we have $16<rs\le48$. Also, the above equation implies $t=\dfrac{16(r+s)}{rs-16}$. Hence we need to find r and s so that $16<rs\le48$ and $rs-16$ divides $16(r+s)$. We can then list out all such pairs of (r, s) and compute the corresponding t, getting 8 different solutions, namely, (r, s, t) = (3, 6, 72), (3, 7, 32), (3, 8, 22), (3, 12, 12), (4, 5, 36), (4, 6, 20), (4, 8, 12) and (6, 7, 8). Therefore there are 8 such triangles.