香港資優教育學院
The Hong Kong Academy for Gifted Education

International Mathematical Olympiad Preliminary Selection Contest - Hong Kong 2011

2011 國際數學奧林匹克 – 香港選拔賽

28 May 2011 (Saturday)
2011 年 5 月 28 日（星期六）

Question Book
問題簿

Instructions to Contestants:
考生須知：

1. The contest comprises a 3-hour written test.
   比賽以筆試形式進行，限時三小時。

2. Questions are in bilingual version. Contestants should answer all questions in either Chinese or English.
   題目中英對照。參賽學生必須解答全卷所有題目，可選擇以全中文或英文作答。

3. Put your answers on the answer sheet.
   請將答案寫在答題紙上。

4. The use of calculators is NOT allowed.
   不可使用計算機。

5. Measuring instruments like rulers, compasses, etc. can be used.
   直尺、圓規及其它量度工具可作輔助之用。

Co-organised by The Hong Kong Academy for Gifted Education,
the Gifted Education Section of the Education Bureau and
International Mathematical Olympiad Hong Kong Committee
香港資優教育學院、教育局資優教育組及國際數學奧林匹克香港委員會合辦

1. Find the sum of all fractions in lowest terms with value greater than 10 but smaller than 100 and with denominator equal to 3. (1 mark)

求所有大於 10 且小於 100，且以 3 為分母的最簡分數之和。（1 分）

2. For positive integer n, let $f(n)$ denote the unit digit of $1+2+3+\cdots+n$. Find the value of $f(1)+f(2)+\cdots+f(2011)$. (1 mark)

對於正整數 n，設 $f(n)$ 表示 $1+2+3+\cdots+n$ 的個位數字。求 $f(1)+f(2)+\cdots+f(2011)$ 的值。（1 分）

3. Find a positive integer k whose product of digits is equal to $\frac{11k}{4}-199$. (1 mark)

求一個正整數 k，使其數字之積等於 $\frac{11k}{4}-199$。（1 分）

4. Find a factor of 1464101210001, such that it lies between 1210000 and 1220000. (1 mark)

求 1464101210001 在 1210000 和 1220000 之間的一個因數。（1 分）

5. The 11 letters of ‘IMOHKPRELIM’ are written on 11 cards (one letter on each card). If three cards are randomly drawn, how many different combinations of the three letters are there? (Order is not considered. For instance ‘IMO’ and ‘IOM’ are regarded to be the same combination.) (1 mark)

現把「IMOHKPRELIM」這 11 個英文字母分別寫在 11 張卡紙上（每張卡紙寫上一個字母）。若隨意抽出三張卡紙，所抽出的三個英文字母有多少個不同組合？（不考慮次序，例如「IMO」和「IOM」視為相同組合。）（1 分）

6. The parabola $y=ax^2+bx+c$ has vertex at $\left(\frac{1}{4},-\frac{9}{8}\right)$. If $a>0$ and $a+b+c$ is an integer, find the minimum possible value of a. (1 mark)

拋物線 $y=ax^2+bx+c$ 的頂點在 $\left(\frac{1}{4},-\frac{9}{8}\right)$。若 $a>0$ 而 $a+b+c$ 為整數，求 a 的最小可能值。（1 分）

7. Four points are randomly chosen from the vertices of a regular 12-sided polygon. Find the probability that the four chosen points form a rectangle (including square). (1 mark)

從一個正十二邊形的頂點中隨意選四點，求該四點成一長方形（包括正方形）的概率。（1 分）

8. In ΔABC, $BC = 5$, $AC = 4$ and $\cos(A-B)=\frac{7}{8}$. Find the value of $\cos C$. (1 mark)

在 ΔABC 中，$BC = 5$、$AC = 4$ 而 $\cos(A-B)=\frac{7}{8}$。求 $\cos C$ 的值。（1 分）

9. On the rectangular coordinate plane, a point P is randomly chosen on the line segment joining (0, –1) and (0, 1). Another point Q is randomly chosen on the line segment joining (1, –1) and (1, 1). Two circles, each with radius 1, are then drawn using P and Q as centres respectively. Find the probability that the two circles intersect. (1 mark)

在直角坐標平面上，我們隨意在連接 (0, –1) 和 (0, 1) 的線段上選一點 P，再隨意在連接 (1, –1) 和 (1, 1) 的線段上選一點 Q，然後分別以 P 和 Q 為圓心，各作一個半徑為 1 的圓。求兩圓相交的概率。（1 分）

10. In ΔABC, $AB = 9$, $BC = 8$ and $AC = 7$. The bisector of $\angle A$ meets BC at D. The circle passing through A and touching BC at D cuts AB and AC at M and N respectively. Find MN. (1 mark)

在 ΔABC 中，$AB = 9$，$BC = 8$ 及 $AC = 7$。$\angle A$ 的角平分線交 BC 於 D。穿過 A 且與 BC 相切於 D 的圓分別交 AB 和 AC 於 M 和 N。求 MN。（1 分）

11. The lengths of the three altitudes of a triangle are in the ratio $3:4:6$. If the radius of the inscribed circle of the triangle is 1, find the radius of the circumcircle of the triangle. (2 marks)

某三角形三條高的長度之比為 $3:4:6$。若該三角形內切圓的半徑是 1，求它的外接圓半徑。（2 分）

12. $ABCD$ is a square of side length 1. P is a point on AC and the circumcircle of ΔBPC cuts CD at Q. If the area of ΔCPQ is $\frac{6}{25}$, find CQ. (2 marks)

$ABCD$ 是個邊長為 1 的正方形。P 是 AC 上的一點，且 ΔBPC 的外接圓交 CD 於 Q。若 ΔCPQ 的面積為 $\frac{6}{25}$，求 CQ。（2 分）

13. If the eight digits 1, 2, 3, 4, 5, 6, 7, 8 are randomly permuted, what is the probability that the resulting eight-digit number is divisible by 8? (2 marks)

若把 1、2、3、4、5、6、7、8 這八個數字任意排列，所得八位數可被 8 整除的概率是多少？（2 分）

14. 2011 cards are arranged in a row on a table. One of the numbers ‘1’, ‘2’ and ‘3’ is printed on each card. It is found that there is at least one card between any two cards labelled ‘1’, at least two cards between any two cards labelled ‘2’, and at least three cards between any two cards labelled ‘3’. If the smallest and greatest possible numbers of cards labelled ‘3’ are m and M respectively, find the value of $m+M$. (2 marks)

桌子上有 2011 張咭，每張均寫上「1」、「2」或「3」。已知任意兩張寫上「1」的咭之間均最少有一張咭，任意兩張寫上「2」的咭之間均最少有兩張咭，任意兩張寫上「3」的咭之間均最少有三張咭。若寫上「3」的咭的數目的最小值和最大值分別是 m 和 M，求 $m+M$ 的值。（2 分）

15. A teacher asked each student of a class to write down the number of classmates with the same surname as his, as well as the number of classmates whose birthday is the same as his. Among the answers received, each of the numbers 0, 1, 2, …, 10 has appeared. Find the minimum number of students in the class. (2 marks)

老師要求某班的每位學生分別寫下班中姓氏和自己相同的同學數目，及班中和自己同一天生日的同學數目。在收到的答案中，0、1、2、⋯、10 各數均有出現。求該班學生人數的最小可能值。（2 分）

16. Let a, b, c be real numbers with $c \neq 1$. It is known that the two equations $x^2+ax+1=0$ and $x^2+bx+c=0$ have a common real root, and so do the two equations $x^2+x+a=0$ and $x^2+cx+b=0$. Find the value of $a+b+c$. (2 marks)

設 a、b、c 為實數，其中 $c \neq 1$。已知 $x^2+ax+1=0$ 和 $x^2+bx+c=0$ 這兩條方程有一個公共實根，而 $x^2+x+a=0$ 和 $x^2+cx+b=0$ 這兩條方程亦有一個公共實根。求 $a+b+c$。（2 分）

17. In ΔABC, $AB=2AC$ and $\angle BAC$ = 112°. P and Q are points on BC such that $AB^2+BC\cdot CP=BC^2$ and $3AC^2+2BC\cdot CQ=BC^2$. Find $\angle PAQ$. (2 marks)

在 ΔABC 中，$AB=2AC$ 且 $\angle BAC$ = 112°。P 和 Q 是 BC 上的點，使得 $AB^2+BC\cdot CP=BC^2$ 及 $3AC^2+2BC\cdot CQ=BC^2$。求 $\angle PAQ$。（2 分）

18. How many ways are there to arrange 10 identical red balls, 5 identical green balls and 5 identical blue balls in a row so that no two adjacent balls are of the same colour? (2 marks)

有多少種方法把 10 個相同的紅球、5 個相同的綠球和 5 個相同的藍球排成一行，使得沒有兩個相鄰的球的顏色相同？（2 分）

19. A bank issues ATM cards to its customers. Each card is associated with a password, which consists of 6 digits with no three consecutive digits being the same. It is known that no two cards have the same password. What is the maximum number of ATM cards the bank has issued? (2 marks)

某銀行發行提款咭供其客戶使用。每張提款咭均有一個六位數字的密碼，且沒有三個連續數字是相同的。已知沒有兩張提款咭的密碼相同。問該銀行最多發行了多少張提款咭？（2 分）

20. Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three different real solutions to the system of equations $\begin{cases} x^3-5xy^2=21 \\ y^3-5x^2y=28 \end{cases}$. Find the value of $\left(11-\frac{x_1}{y_1}\right)\left(11-\frac{x_2}{y_2}\right)\left(11-\frac{x_3}{y_3}\right)$. (2 marks)

設 (x_1, y_1)、(x_2, y_2) 和 (x_3, y_3) 分別為方程組 $\begin{cases} x^3-5xy^2=21 \\ y^3-5x^2y=28 \end{cases}$ 三個不同的實數解。求 $\left(11-\frac{x_1}{y_1}\right)\left(11-\frac{x_2}{y_2}\right)\left(11-\frac{x_3}{y_3}\right)$ 的值。（2 分）

– End of Paper 全卷完 –