International Mathematical Olympiad

Preliminary Selection Contest 2009 — Hong Kong

Outline of Solutions

Answers:

1.	4038091	**2.**	$102°$	**3.**	40	**4.**	3335
5.	$\dfrac{4040099}{2010}$	**6.**	$\dfrac{37}{5}$	**7.**	$4-\sqrt{13}$	**8.**	$\dfrac{3-\sqrt{5}}{2}$
9.	$\dfrac{3}{4}$	**10.**	$\dfrac{8}{35}$	**11.**	$\dfrac{2009}{2010}$	**12.**	63
13.	3	**14.**	89	**15.**	8074171	**16.**	1000
17.	230	**18.**	$\sqrt{6}$	**19.**	$\sqrt{2}$	**20.**	11457

Solutions:

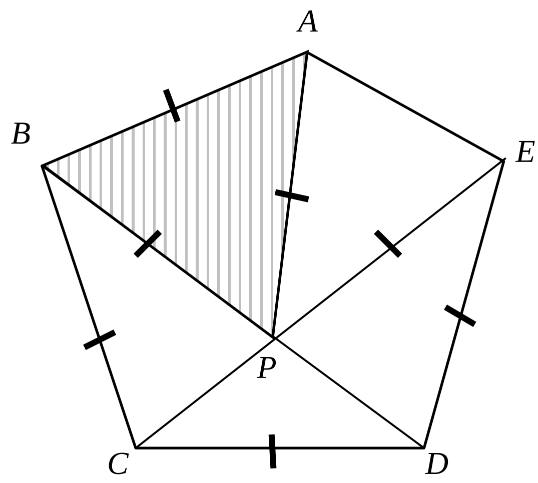
1. We have

$$\begin{aligned}\frac{1^4+a^4+(a+1)^4}{1^2+a^2+(a+1)^2} &= \frac{1^4+a^4+a^4+4a^3+6a^2+4a+1}{1^2+a^2+a^2+2a+1} \\ &= \frac{2(a^4+2a^3+3a^2+2a+1)}{2(a^2+a+1)} \\ &= \frac{(a^2+a+1)^2}{a^2+a+1} \\ &= a^2+a+1\end{aligned}$$

and so setting $a=2009$ gives the answer $2009^2+2009+1=4038091$.

Remark. It would be easier to start by plugging in small values of a and then try to look for a pattern.

2. Let P be the intersection of BD and CE. Then $\angle PBC = \angle PED = \angle PCD = \angle PDC = \dfrac{180°-108°}{2} = 36°$, and so $\angle BCP = \angle BPC = 72°$. Hence $BP = BC = AB$ and $\angle ABP = 96° - 36° = 60°$. It follows that ΔABP is equilateral and $AP = PE$. Thus $\angle APE = 108^{o} - 60° = 48°$ and $\angle AEP = \dfrac{180°-48°}{2} = 66°$, and so $\angle E = \angle AED = 66° + 36° = 102°$.



3. Suppose Donald, Henry and John started with $14n$, $5n$ and n candies respectively. After Donald gave John 20 candies and Henry shared candies with John, they had $14n-20$, $3n+10$ and $3n+10$ candies respectively. Hence we have $14n-20=3(3n+10)$, or $n=10$. In the end, Donald had $14n-20+x=120+x$ candies while Henry had $3n+10+x=40+x$ candies. Hence we have $120+x=2(40+x)$, giving $x=40$.

4. Note that $9997n=(10000-3)n=10000(n-1)+(10000-3n)$. Thus n must be odd to ensure that the last digit of $9997n$ is odd, and hence $n-1$ is even. If $n<3335$, then $0<10000-3n<10000$ and hence the ten thousands digit of $9997n$ is the same as the unit digit of $10000(n-1)$, which is even. Therefore $9997n$ contains an even digit. If $n=3335$, we find that $9997\times 3335=33339995$. It follows that the smallest possible value of n is 3335.

5. For $n\geq 2$, we have

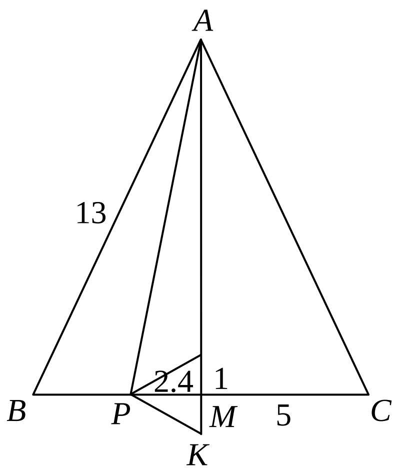
$$\begin{aligned}
x_n{}^2 &= x_1{}^2+\sum_{k=1}^{n-1}(x_{k+1}{}^2-x_k{}^2) \\
&= x_1{}^2+\sum_{k=1}^{n-1}\left[\frac{1}{(k+2)^2}-\frac{1}{k^2}\right] \\
&= \left(\frac{3}{2}\right)^2+\frac{1}{(n+1)^2}+\frac{1}{n^2}-\frac{1}{2^2}-\frac{1}{1^2} \\
&= \left(\frac{1}{n}-\frac{1}{n+1}+1\right)^2
\end{aligned}$$

Since $x_n>0$, we have $x_n=\dfrac{1}{n}-\dfrac{1}{n+1}+1$ (note that this also holds for $n=1$). Hence

$$\sum_{n=1}^{2009}x_n=\sum_{n=1}^{2009}\left(\frac{1}{n}-\frac{1}{n+1}+1\right)=1-\frac{1}{2010}+2009=\frac{4040099}{2010}.$$

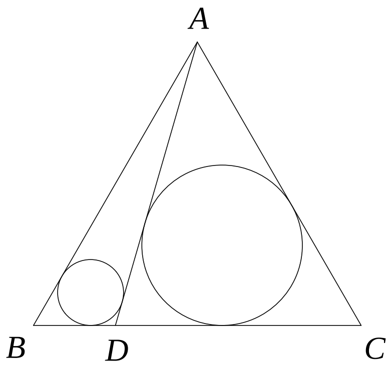
Remark. One could easily see a pattern and make a guess for the answer by computing x_1, x_1+x_2, $x_1+x_2+x_3$ and so on.

6. Note that AHK is perpendicular to BC, PH is perpendicular to AC and PK is perpendicular to AB. Hence it is easy to see that $\Delta PHK \sim \Delta ABC$. Let M be the midpoint of BC (which is also the midpoint of HK). Then $PM = AM \times \dfrac{HK}{BC} = \sqrt{13^2-5^2} \times \dfrac{2}{10} = 2.4$, and so $PC = PM + MC = 2.4 + 5 = 7.4$.



7. We have $AD = \sqrt{4^2+1^2-2(4)(1)\cos 60^\circ} = \sqrt{13}$. The area of ΔADB is $\dfrac{1}{2}(4)(1)\sin 60^\circ = \sqrt{3}$, and similarly ΔADC has area $3\sqrt{3}$. Recall that the radius of the inscribed circle of a triangle is equal to twice the area divided by the perimeter. (This can be seen by connecting the in-centre to the three vertices and then considering area.) It follows that

$$rs = \frac{2\cdot\sqrt{3}}{4+1+\sqrt{13}} \cdot \frac{2\cdot 3\sqrt{3}}{4+3+\sqrt{13}} = \frac{36}{48+12\sqrt{13}} = 4-\sqrt{13}.$$



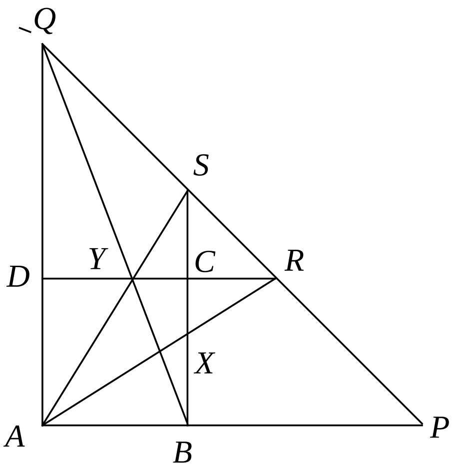
8. Let $CR = CS = n$ and $DQ = k$.

Since $\Delta RCX \sim \Delta RDA$, we have $\dfrac{n}{m} = \dfrac{n+1}{1}$ or $n = \dfrac{m}{1-m}$.

Since $\Delta QDY \sim \Delta QAB$, we have $\dfrac{k}{1-m} = \dfrac{k+1}{1}$ or $k = \dfrac{1-m}{m}$.

Since $\Delta RCS \sim \Delta RDQ$, we have $\dfrac{n}{k} = \dfrac{n}{n+1}$ or $k = n+1$.

From the above, we get $\dfrac{1-m}{m} = \dfrac{m}{1-m} + 1$. Solving this equation subject to the condition $0 < m < 1$, we get $m = \dfrac{3-\sqrt{5}}{2}$.



9. Note that on the line segment joining (a, b) and (c, d), where a, b, c, d are integers, the number of lattice points (excluding the endpoints) is exactly 1 less than the H.C.F. of $c-a$ and $d-b$.

Hence there is an even number of lattice points on the line segment (excluding the endpoints) joining (0, 2010) and (a, b) if and only if the H.C.F. of a and $2010-b$ is odd, which is true if and only if at least one of a, b is odd.

Now there are 100 choices for each of a and b (namely, 1, 2 , …, 100). Hence the probability that both are even is $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$, and so the required probability is $1-\frac{1}{4}=\frac{3}{4}$.

10. Randomly arrange the players in a row (this can be done in 8! ways) so that the two leftmost players compete against each other, the next two compete against each other and so on. Now suppose we want no two mathematicians to play against each other. Then there are 8 possible positions for the first mathematician, then 6 position for the second mathematician, and similarly 4 for the third and 2 for the last. After the positions of the mathematicians are fixed, there are 4! ways to arrange the positions of the non-mathematicians. Hence the answer is $\frac{8\times6\times4\times2\times4!}{8!}=\frac{8}{35}$.

11. Note that $\tan^{-1}\frac{1}{2k^2}=\tan^{-1}\frac{(2k+1)-(2k-1)}{1+(2k-1)(2k+1)}=\tan^{-1}(2k+1)-\tan^{-1}(2k-1)$. Hence

$$\begin{aligned}\tan\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{2\times2^2}+\tan^{-1}\frac{1}{2\times3^2}+\cdots+\tan^{-1}\frac{1}{2\times2009^2}\right)&=\tan(\tan^{-1}(2\times2009+1)-\tan^{-1}1)\\&=\tan\left(\tan^{-1}\frac{2\times2009+1-1}{1+(2\times2009+1)}\right)\\&=\frac{2009}{2010}\end{aligned}$$

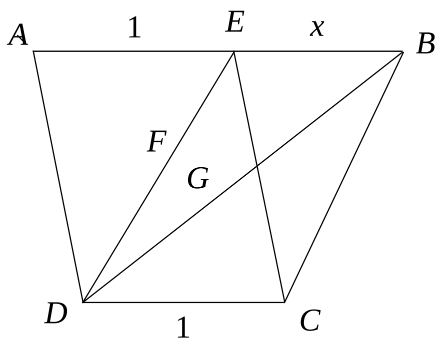
Remark. One could easily see a pattern and make a guess for the answer by computing $\tan\left(\tan^{-1}\frac{1}{2}\right)$, $\tan\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{2\times2^2}\right)$, $\tan\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{2\times2^2}+\tan^{-1}\frac{1}{2\times3^2}\right)$ and so on.

12. Colour the cells by black and white alternately. It is clear that the numbers of coins in cells of same colour are of the same parity and the numbers of coins in cells of different colours are of different parity. Hence n must be odd, for otherwise the total number of coins must be even since there are an even number of white cells and an even number of black cells.

Now we may assume there are $\frac{n^2+1}{2}$ black cells and $\frac{n^2-1}{2}$ white cells. Then we must have

$\frac{n^2+1}{2} \le 2009$ and $\frac{n^2-1}{2} \le 2009$. Hence $n \le 63$. It is possible for $n = 63$, by putting one coin in each of the $\frac{63^2+1}{2} = 1985$ black cells, we then put two coins on each of the first 12 white cells of the first row. Together we have $1985 + 12 \times 2 = 2009$ coins on the chessboard. It follows that the answer is 63.

13. We denote by $[XYZ]$ the area of XYZ. By scaling we may assume $AE = DC = 1$ and $EB = x$. Since $\Delta CDG \sim \Delta ABG$, we have $\frac{CG}{AG} = \frac{1}{1+x}$. Note that $AECD$ is a parallelogram, so $AF = FC$ and hence



$$\frac{FG}{AC} = \frac{\frac{1}{2}[1+(1+x)]-1}{1+(1+x)} = \frac{x}{4+2x}.$$

From this we get

$$\frac{[DFG]}{[ABCD]} = \frac{\frac{FG}{AC} \cdot [ACD]}{2[ACE] + x[ACE]} = \frac{x}{(4+2x)(2+x)} = \frac{1}{2\left(\frac{2}{\sqrt{x}} + \sqrt{x}\right)^2} = \frac{1}{2\left(\frac{2}{\sqrt{x}} - \sqrt{x}\right)^2 + 8}$$

This is maximum when $\frac{2}{\sqrt{x}} = \sqrt{x}$, or $x = 2$, in which case $\frac{AB}{CD} = 1 + x = 3$.

14. Let m be the number of people taking the quiz. Then we have

$$\begin{aligned} 2009 &= n + (n-2) + (n-4) + \cdots + (n-2m+2) \\ &= \frac{m}{2}\left[n + (n-2m+2)\right] \\ &= m(n-m+1) \end{aligned}$$

Hence $n = \frac{2009}{m} + m - 1$, where m is a factor of 2009. We check that $m = 41$ gives the smallest value of n, which is 89.

15. Note that $\sqrt{2009^2+m} - 2009 = \frac{m}{\sqrt{2009^2+m} + 2009}$. For $1 \le m \le 4018$, we have

$$\frac{m}{4019} < \frac{m}{\sqrt{2009^2+m} + 2009} < \frac{m}{4018}.$$

Hence each summand is equal to 2009 plus a 'decimal part'. Summing up all the 'decimal parts', we have

$$2009=\sum_{m=1}^{4018}\frac{m}{4019}<\left(\sum_{m=1}^{4018}\sqrt{2009^2+m}\right)-2009\times 4018<\sum_{m=1}^{4018}\frac{m}{4018}=2009\frac{1}{2}$$

and so the answer is $2009\times 4018+2009=8074171$.

16. Note that if (a, b, c) is a positive integer solution to the equation $4x+3y+2z=2009$, then $(a-1,b-1,c-1)$ is a *non-negative* integer solution to the equation $4x+3y+2z=2000$. Conversely, every positive integer solution to $4x+3y+2z=2000$ corresponds to a positive integer solution to $4x+3y+2z=2009$.

    Hence positive integer solutions to the two equations are in one-to-one correspondence except for solutions to $4x+3y+2z=2009$ with at least one variable equal to 1, in which case the corresponding solutions to $4x+3y+2z=2000$ have at least one variable being 0. The number of such solutions is exactly equal to the value of $f(2009)-f(2000)$. To count this number, we note that

    - if $x=1$, the equation becomes $3y+2z=2005$ and so y may be equal to 1, 3, 5, …, 667 so that z is a positive integer, i.e. there are $\frac{667-1}{2}+1=334$ solutions in this case;
    - if $y=1$, the equation becomes $4x+2z=2006$, or $2x+z=1003$, which has 501 positive integer solutions (corresponding to x = 1, 2, …, 501); and
    - if $z=1$, the equation becomes $4x+3y=2007$ and so y may be equal to 1, 5, 9, …, 665 so that z is a positive integer, i.e. there are $\frac{665-1}{4}+1=167$ solutions in this case.

    Two of these solutions have been double-counted (corresponding to $x=y=1$ and $y=z=1$). It follows that the answer is $334+501+167-2=1000$.

17. Note that at least three colours have to be used. So we have four possibilities.

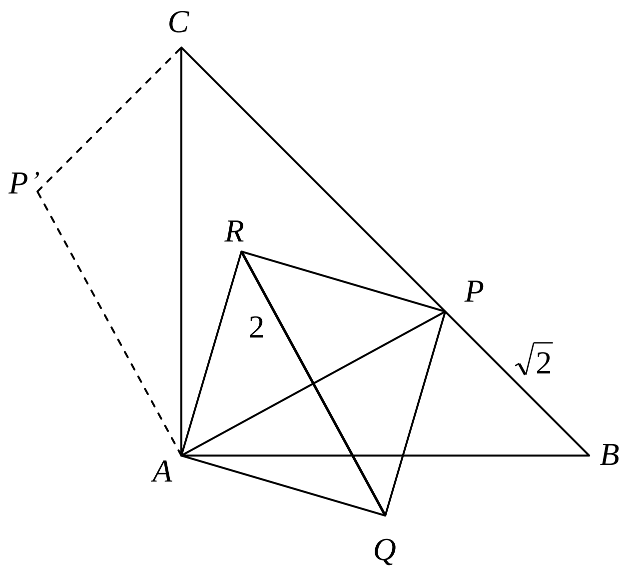
    - If exactly three colours are used, there are $C_3^6=20$ ways to choose colours. Then each pair of opposite faces must be painted in the one colour, and there is only one way of colouring (up to rotation). Hence there are 20 colourings in this case.
    - If exactly four colours are used, there are $C_4^6=15$ ways to choose colours. There must be exactly one pair of opposite faces with different colours, and this pair can be coloured in $C_2^4=6$ ways. After that the colours of the other four faces are fixed (up to rotation) as each pair of opposite faces must receive one colour. Hence there are $15\times 6=90$ colourings in this case.
    - If exactly five colours are used, there are $C_5^6=6$ ways to choose colours. There must be exactly one pair of opposite faces with the same colour, and there are 5 choices of colours for this pair (say, top and bottom faces). After that, there are four faces in a ring shape to

be assigned the remaining four colours. We only have to split the four colours into two pairs so that each pair of colours belongs to a pair of adjacent faces. There are 3 ways to do so (just fix one colour and consider which colour to pair up). Hence there are $6\times5\times3=90$ colourings in this case.

- If all six colours are used, fix any colour and assume (rotate suitably if necessary) that it is for the bottom face. Then there are 5 choices for the colour of the top face. There are now 6 ways to assign the remaining four colours to the four lateral faces. (Not 3 ways as in the previous case, because the top and bottom faces now have different colours, so switching the colours of a pair of opposite lateral faces results in a different colouring.) Hence there are $5\times6=30$ colourings in this case.

Combining the four cases, the answer is $20+90+90+30=230$.

18. Let *P'* be the image of *P* by an anticlockwise rotation about *A* through 90° so that *B* goes to *C*. As $\angle AP'C + \angle APC = \angle APB + \angle APC = 180°$, *APCP'* is a cyclic quadrilateral. So the circumcentre *R* is the image of the circumcentre *Q* under the rotation. Thus *AQR* is also a right-angled isosceles triangle, and so *AQPR* is a square and *AP* = *QR* = 2. Let *AB* = *x*. Applying the cosine law in Δ*ABP*, we have



$$x^2+(\sqrt{2})^2-2x(\sqrt{2})\cos 45°=2^2.$$

This gives $x^2-2x-2=0$ and hence $x=1+\sqrt{3}$. It follows that $PC=\sqrt{2}$ and $AB-BP=\sqrt{2}+\sqrt{6}-\sqrt{2}=\sqrt{6}$

19. Expressing *w*, *z* and *y* in terms of *t* and *x*, we have

$$w=\frac{xt-1}{x},\quad z=\frac{xt^2-t-x}{xt-1}\quad\text{and}\quad y=\frac{xt^3-t^2-2xt+1}{xt^2-t-x}.$$

Substituting the last expression into $x+\frac{1}{y}=t$, we have

$$x+\frac{xt^2-t-x}{xt^3-t^2-2xt+1}=t$$
$$xt^4-x^2t^3-t^3-2xt^2+2x^2t+2t=0$$
$$t(t^2-2)(xt-x^2-1)=0$$

Note that the last factor on the left hand side of the last row is non-zero, for otherwise we get

$x+\frac{1}{x}=t=x+\frac{1}{y}$ which contradicts the fact that $x \neq y$. Since $t>0$, we must have $t=\sqrt{2}$.

Remark. Owing to some typo, $\sqrt{2}$, $-\sqrt{2}$ and $\pm\sqrt{2}$ were all accepted as correct answers in the live paper.

20. Note that each of *a*, *b*, *c*, *d* is of the form $3^m 7^n$ where *m* is one of 0, 1, 2, 3 and *n* is one of 0, 1, 2, 3, 4, 5. Furthermore, at least two of the four *m*'s are equal to 3 and at least two of the four *n*'s are equal to 5.

    We first count the number of ways of choosing the four *m*'s. There are three possibilities:

    - If all four of them are 3, there is 1 choice.
    - If exactly three of them are 3, there are 4 ways to choose the outlier and 3 ways to choose the value of the outlier (namely, 0, 1 or 2), giving $4\times 3=12$ choices.
    - If exactly two of them are 3, there are $C_2^4=6$ ways to locate the 3's, then $3^2=9$ ways to choose the value of the other two *m*'s, giving $6\times 9=54$ choices.

    Hence there are $1+12+54=67$ ways to choose the values of the *m*'s. Likewise, there are $1+4\times 5+C_2^4\times 5^2=171$ ways to choose the values of the *n*'s. It follows that the answer is $67\times 171=11457$.