International Mathematical Olympiad
Hong Kong Preliminary Selection Contest 2007

國際數學奧林匹克
香港選拔賽初賽 2007

26th May 2007

2007 年 5 月 26 日

Time allowed: 3 hours

時限：3 小時

Instructions to Candidates:

考生須知：

1. Answer ALL questions.

   本卷各題全答。

2. Put your answers on the answer sheet.

   請將答案寫在答題紙上。

3. The use of calculators is NOT allowed.

   不可使用計算機。

Section A (1 mark each)
甲部（每題 1 分）

1. Someone forms an integer by writing the integers from 1 to 82 in ascending order, i.e. 1234567891011…808182. Find the sum of the digits of this integer.
   某人把 1 到 82 之間的整數順序寫出來，從而得到另一個整數 1234567891011…808182。求這個整數的數字之和。

2. Find the smallest positive integer n for which the last three digits of $2007n$ (in decimal notation) are 837.
   求最小的正整數 n，使得 $2007n$（在十進制中）的最後三位數字是 837。

3. $ABCD$ is a parallelogram with $\angle D$ obtuse. M, N are the feet of perpendiculars from D to AB and BC respectively. If $DB = DC = 50$ and $DA = 60$, find $DM + DN$.
   $ABCD$ 是平行四邊形，其中 D 是鈍角。M 和 N 分別爲 D 到 AB 和 BC 的垂足。若 $DB = DC = 50$ 而 $DA = 60$，求 $DM + DN$。

4. One day, a truck driver drove through a tunnel and measured the time taken between the moment at which the truck started entering the tunnel and the moment at which the truck left the tunnel completely. The next day a container was added and the length of the truck was increased from 6 m to 12 m. The driver reduced the speed by 20% and measured the time again. He found that the time taken was increased by half. Find the length of the tunnel (in metres).
   一位貨櫃車司機駕車穿過隧道時，計算了車子進入隧道直至整輛車離開隧道所需的時間。第二天，車子加上了一個貨櫃，使其總長度由 6 米變成 12 米。司機把車速調低 20%，並再次計算同一時間。他發現需時增加了一半。求隧道的長度（以米爲單位）。

5. ΔABC has area 1. E and F are points on AB and AC respectively such that $EF \parallel BC$. If ΔAEF and ΔEBC have equal area, find the area of ΔEFC.
   ΔABC 的面積是 1。E、F 分別是 AB 和 AC 上的點，使得 $EF \parallel BC$。若 ΔAEF 和 ΔEBC 的面積相等，求 ΔEFC 的面積。

6. Let $[x]$ denote the greatest integer not exceeding x. If p, q, r are positive, find the minimum value of:
   設 $[x]$ 表示不超過 x 的最大整數。若 p、q、r 爲正數，求下式的最小值：
   $$\left[\frac{p+q}{r}\right]+\left[\frac{q+r}{p}\right]+\left[\frac{r+p}{q}\right]$$

7. Let $n = \underbrace{999\ldots999}_{2007\text{ digits}}$. How many '9's are there in the decimal representation of n^3?
   設 $n = \underbrace{999\ldots999}_{2007\text{位}}$。以十進制表示 n^3 時，數字「9」共會出現多少次？

8. Let x, y be nonnegative integers such that $x+2y$ is a multiple of 5, $x+y$ is a multiple of 3 and $2x+y \geq 99$. Find the minimum possible value of $7x+5y$.
   設 x、y 爲非負整數，使得 $x+2y$ 爲 5 的倍數、$x+y$ 爲 3 的倍數，且 $2x+y \geq 99$。求 $7x+5y$ 的最小可能值。

9. The three-digit number $\overline{abc}$ consists of three non-zero digits. The sum of the other five three-digit numbers formed by rearranging a, b, c is 2017. Find $\overline{abc}$.
三位數 $\overline{abc}$ 由三個非零數字組成。若把 a、b、c 重新排列，則其餘五個可組成的三位數之和是 2017。求 $\overline{abc}$。

10. Find the sum of the greatest odd factor of each of the numbers 2007, 2008, …, 4012.
求 2007、2008、…、4012 各數的最大奇因數之和。

11. Let A_1, A_2, …, A_{11} be 11 points on a straight line in order, where $A_1A_{11} = 56$. Given that $A_iA_{i+2} \le 12$ for $i = 1, 2, \ldots, 9$ and $A_jA_{j+3} \ge 17$ for $j = 1, 2, \ldots, 8$, find A_2A_7.
設 A_1、A_2、…、A_{11} 爲一條直線上順序的 11 點，其中 $A_1A_{11} = 56$。已知對於 $i = 1, 2, \ldots, 9$ 皆有 $A_iA_{i+2} \le 12$，且對 $j = 1, 2, \ldots, 8$ 皆有 $A_jA_{j+3} \ge 17$。求 A_2A_7。

12. In ΔABC, $AB = \sqrt{5}$, $BC = 1$ and $AC = 2$. I is the incentre of ΔABC and the circumcircle of ΔIBC cuts AB at P. Find BP.
在 ΔABC 中，$AB = \sqrt{5}$、$BC = 1$、$AC = 2$。I 是 ΔABC 的內心，且 ΔIBC 的外接圓交 AB 於 P。求 BP。

13. Let x_1, x_2, x_3, x_4, x_5 be nonnegative real numbers whose sum is 300. Let M be the maximum of the four numbers $x_1 + x_2$, $x_2 + x_3$, $x_3 + x_4$ and $x_4 + x_5$. Find the least possible value of M.
設 x_1、x_2、x_3、x_4、x_5 爲非負實數，且它們之和爲 300。以 M 表示 $x_1 + x_2$、$x_2 + x_3$、$x_3 + x_4$ 和 $x_4 + x_5$ 四數中最大的一個。求 M 的最小可能值。

14. $ABCD$ is a square with side length 9. Let P be a point on AB such that $AP : PB = 7 : 2$. Using C as centre and CB as radius, a quarter circle is drawn inside the square. The tangent from P meets the circle at E and AD at Q. The segments CE and DB meet at K, while AK and PQ meet at M. Find the length of AM.
$ABCD$ 是一個邊長爲 9 的正方形。設 P 爲 AB 上的一點，使得 $AP : PB = 7 : 2$。以 C 爲圓心、CB 爲半徑在正方形內作一個四分一圓，從 P 點到圓的切線與圓相交於 E，與 AD 相交於 Q。CE 和 DB 交於 K，而 AK 和 PQ 則交於 M。求 AM 的長度。

15. $ABCD$ is a rectangle with $AB = 2$ and $BC = 1$. A point P is randomly selected on CD. Find the probability that $\angle APB$ is the largest among the three interior angles of ΔPAB.
$ABCD$ 是長方形，其中 $AB = 2$，$BC = 1$。現於 CD 上隨意選一點 P。求 $\angle APB$ 是 ΔPAB 三隻內角中最大的一隻的概率。

16. Let a, b, c be positive integers such that $ab + bc - ca = 0$ and $a - c = 101$. Find b.
設 a、b、c 爲正整數，其中 $ab + bc - ca = 0$ 而 $a - c = 101$。求 b。

17. A bag contains 15 balls, marked with the 15 numbers 2^0, 2^1, 2^2, …, 2^{14} respectively. Each ball is either red or blue, and there is at least one ball of each colour. Let a be the sum of the numbers on all red balls, b be the sum of the numbers on all blue balls and d be the H.C.F. of a and b. Find the greatest possible value of d.
一個袋子中有 15 個球，它們分別寫上 2^0、2^1、2^2、…、2^{14} 這 15 個數。每個球都是紅色或藍色的，而且紅球和藍球均最少各有一個。設 a 爲所有紅球上的各數之和、b 爲所有藍球上的各數之和、d 爲 a 和 b 的最大公因數。求 d 的最大可能值。

18. Find the sum of all real roots of the equation $3\tan^2 x + 8\tan x + 3 = 0$ in the range $0 < x < 2\pi$.
求方程 $3\tan^2 x + 8\tan x + 3 = 0$ 在區間 $0 < x < 2\pi$ 內所有實根之和。

19. For $0 \le x \le 1$ and positive integer n, let $f_0(x) = |1-2x|$ and $f_n(x) = f_0(f_{n-1}(x))$. How many solutions are there to the equation $f_{10}(x) = x$ in the range $0 \le x \le 1$?
對於 $0 \le x \le 1$ 和正整數 n，設 $f_0(x) = |1-2x|$ 和 $f_n(x) = f_0(f_{n-1}(x))$。那麼方程 $f_{10}(x) = x$ 在 $0 \le x \le 1$ 區間內有多少個解？

20. Determine the number of ordered pairs (x, y) of positive integers satisfying the following equation:
求滿足下式的正整數序偶對 (x, y) 的數目：

$$x\sqrt{y} + y\sqrt{x} - \sqrt{2007x} - \sqrt{2007y} + \sqrt{2007xy} = 2007$$

Section B (2 marks each)
乙部（每題 2 分）

21. How many nine-digit positive integers consist of nine pairwise distinct digits and are divisible by 4950?
有多少個九位正整數的九個數字互不相同，而且可被 4950 整除？

22. Let S be the set of points whose coordinates x, y and z are integers that satisfy $0 \le x \le 2$, $0 \le y \le 3$ and $0 \le z \le 4$. Two distinct points are randomly chosen from S. Find the probability that the midpoint of the two chosen points also belongs to S?
設 S 爲座標 x、y、z 皆爲整數，且滿足 $0 \le x \le 2$、$0 \le y \le 3$ 和 $0 \le z \le 4$ 的點集。現從 S 中隨意抽出兩個不同的點，求該兩點的中點亦屬於 S 的概率。

23. If x is positive, find the minimum value of $\dfrac{\sqrt{x^4 + x^2 + 2x + 1} + \sqrt{x^4 - 2x^3 + 5x^2 - 4x + 1}}{x}$.
若 x 爲正數，求 $\dfrac{\sqrt{x^4 + x^2 + 2x + 1} + \sqrt{x^4 - 2x^3 + 5x^2 - 4x + 1}}{x}$ 的最小值。

24. A bag contains 999 balls, marked with the 999 numbers $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{1000}$ respectively. Each ball is either red or blue, and the number of red balls is a positive even number. Let S denote the product of the numbers on all red balls. Find the sum of all possible values of S.
一個袋子中有 999 個球，它們分別寫上 $\frac{1}{2}$、$\frac{1}{3}$、$\frac{1}{4}$、…、$\frac{1}{1000}$ 這 999 個數。每個球都是紅色或藍色的，其中紅球的數目是正偶數。設 S 爲所有紅球上的數之積。求 S 所有可能值之和。

25. Find the minimum value of $|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$ for any real number x.
對於實數 x，求 $|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$ 的最小值。

– END OF PAPER – 全卷完 –