International Mathematical Olympiad

Preliminary Selection Contest 2005 — Hong Kong

Outline of Solutions

Answers:

1. 5	**2.** $\frac{35}{12}$	**3.** $\frac{5050}{10101}$	**4.** 165
5. 516	**6.** 957	**7.** 72	**8.** 5
9. $\frac{9\sqrt{3}}{32}$	**10.** 6	**11.** 340	**12.** $\frac{\sqrt{14}}{4}$
13. 546	**14.** 333	**15.** $\frac{\sqrt{21}}{3}$	**16.** 101
17. –4659	**18.** $\frac{\sqrt{5}-1}{2}$	**19.** $\frac{4022030}{3}$	**20.** $\frac{2005\sqrt{2006}}{4012}$

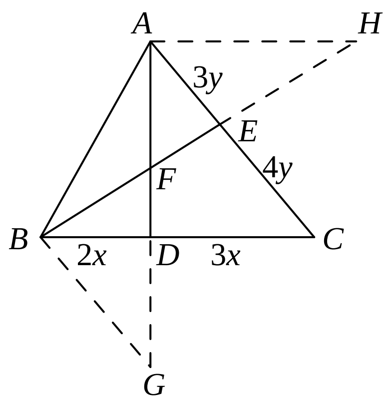
Solutions:

1. Since 2005 is odd, all a_i ’s must be odd. Since the odd a_i ’s add up to 2005, n must be odd as well. Consider the case $n=3$ with $a_1 \ge a_2 \ge a_3$. Then $a_1 \ge \frac{2005}{3}$ and this forces $a_1 = 2005$ by considering the factors of 2005. Then we must have $a_2 + a_3 = 0$ and $a_2 a_3 = 1$, which means $a_2{}^2 + 1 = 0$ and hence leads to no solution. Finally, we see that $n=5$ is possible since

$$2005 + 1 + (-1) + 1 + (-1) = 2005 \times 1 \times (-1) \times 1 \times (-1) = 2005.$$

Hence the answer is 5.

2. Produce AD to G and BE to H such that $AC // BG$ and $BC // AH$. Let $BD = 2x$, $DC = 3x$, $AE = 3y$, $EC = 4y$. Since $\Delta ACD \sim \Delta GBD$, we have $\frac{AC}{BG} = \frac{AD}{DG} = \frac{CD}{DB} = \frac{3}{2}$. Hence $BG = \frac{14y}{3}$ and $AD : DG = 3 : 2$, i.e. $AD = \frac{3}{5} AG$. On the other hand,



since $\Delta AEF \sim \Delta GBF$, $\dfrac{AF}{FG}=\dfrac{FE}{BF}=\dfrac{AE}{BG}=\dfrac{3y}{\frac{14y}{3}}=\dfrac{9}{14}$.

Thus $\dfrac{BF}{FE}=\dfrac{14}{9}$ and $AF=\dfrac{9}{23}FG$. If follows that $FD=AD-AF=\dfrac{3}{5}AG-\dfrac{9}{23}FG=\dfrac{24}{115}FG$.

Thus $\dfrac{AF}{FD}=\dfrac{\frac{9}{23}FG}{\frac{24}{115}FG}=\dfrac{15}{8}$ and hence $\dfrac{AF}{FD}\times\dfrac{BF}{FE}=\dfrac{15}{8}\times\dfrac{14}{9}=\dfrac{35}{12}$.

3. Observe that $1+k^2+k^4=(1+k^2)^2-k^2=(1-k+k^2)(1+k+k^2)$. Let

$$\frac{k}{1+k^2+k^4}\equiv\frac{A}{k(k-1)+1}+\frac{B}{k(k+1)+1}.$$

Solving, we have $A=\dfrac{1}{2}$ and $B=-\dfrac{1}{2}$. It follows that

$$\begin{aligned}&\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+\cdots+\frac{100}{1+100^2+100^4}\\&=\frac{1}{2}\left[\left(\frac{1}{0\times1+1}-\frac{1}{1\times2+1}\right)+\left(\frac{1}{1\times2+1}-\frac{1}{2\times3+1}\right)+\cdots+\left(\frac{1}{99\times100+1}-\frac{1}{100\times101+1}\right)\right]\\&=\frac{1}{2}\left[1-\frac{1}{10101}\right]\\&=\frac{5050}{10101}\end{aligned}$$

4. Let the length and width of the banner be x m and y m respectively. According to the question, x, y are positive integers with $330x+450y\le10000$. Thus the area of the banner is

$$xy=\frac{(330x)(450y)}{(330)(450)}\le\frac{1}{(330)(450)}\left[\frac{330x+450y}{2}\right]^2\le\frac{1}{(330)(450)}\left[\frac{10000}{2}\right]^2<169\,\text{m}^2$$

by the AM-GM inequality. Note that we must have $x\le30$ and $y\le22$.

If the area is 168 m^2, then we have the possibilities (x, y) = (12, 14); (14, 12) and (24, 7), yet none of them satisfies $330x+450y\le10000$. Since 167 is prime and the only factors of 166 are 1, 2, 83 and 166, we easily see that the area cannot be 167 m^2 nor 166 m^2 either.

Finally, we see that $x=15$ and $y=11$ satisfies all conditions and give an area of 165 m^2. Thus the maximum area of the banner is 165 m^2.

5. There are 16 ordered pairs (x, y) of integers satisfying $1\le x\le4$ and $1\le y\le4$. Thus $C_3^{16}=560$ triangles can be formed. We must, however, delete those degenerate triangles, i.e. sets of three

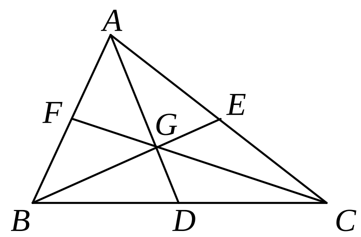
points which are collinear. There are 4 horizontal lines of 4 points and 4 vertical lines of 4 points. These together produce $(4+4)\times C_3^4 = 32$ degenerate triangles. Also, the points (1, 1); (2, 2); (3, 3) and (4, 4) are collinear, leading to $C_3^4 = 4$ degenerate triangles, and so are the points (1, 4); (2, 3); (3, 2) and (4, 1). Finally, we also need to delete the 4 degenerate triangles {(1, 2); (2, 3); (3, 4)}, {(2, 1); (3, 2); (4, 3)}, {(1, 3); (2, 2); (3, 1)} and {(2, 4); (3, 3); (4, 2)}. It follows that the answer is $560-32-4-4-4=516$.

6. Note that $\dfrac{m+4}{m^2+7}$ is in lowest term if and only of $\dfrac{m^2+7}{m+4}$ is in lowest term. Since

$$\frac{m^2+7}{m+4}=\frac{m^2-16}{m+4}+\frac{23}{m+4}=m-4+\frac{23}{m+4},$$

the fraction is in lowest term except when $m+4$ is a multiple of 23. Since m may be equal to 1, 2, …, 1000, we shall count the number of multiples of 23 from 5 to 1004. The first one is $23=23\times 1$ and the last one is $989=23\times 43$. Hence there are 43 such multiples. It follows that the answer is $1000-43=957$.

7. As shown in the figure, let the medians AD, BE and CF of ΔABC meet at the centroid G. Recall that the medians divide ΔABC into 6 smaller triangles of equal area and that the centroid divides each median in the ratio 2 : 1.



Let $AD=9$ and $BE=12$. Then $AG=9\times\dfrac{2}{3}=6$ and $BG=12\times\dfrac{2}{3}=8$. Hence the area of ΔABG is

$$\frac{1}{2}\times AG\times BG\times\sin\angle AGB=\frac{1}{2}\times 6\times 8\times\sin\angle AGB=24\sin\angle AGB\le 24,$$

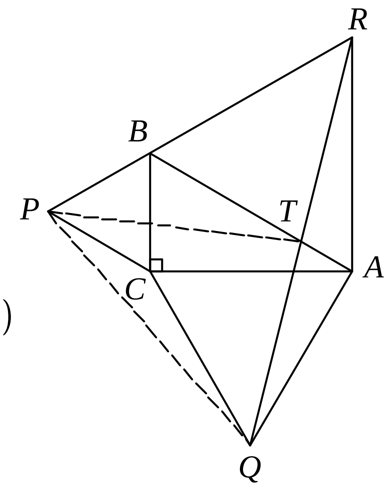
where equality is possible when $\angle AGB=90^\circ$. Since the area of ΔABG is $\dfrac{1}{3}$ the area of ΔABC, the largest possible area of ΔABC is $24\times 3=72$.

8. We have

$$2xy-3x-5y=k$$
$$xy-\frac{3}{2}x-\frac{5}{2}y=\frac{k}{2}$$
$$\left(x-\frac{5}{2}\right)\left(y-\frac{3}{2}\right)=\frac{k}{2}+\frac{15}{4}$$
$$(2x-5)(2y-3)=2k+15$$

When k is a positive integer, the number of positive integral solutions to the above equation is precisely the number of positive divisors of $2k+15$. For this number to be odd, $2k+15$ has to be a perfect square. The smallest such k is 5. Indeed, we can check that when $k = 5$, the original equation has 3 solutions, namely, (3, 14); (5, 4) and (15, 2).

9. Note that $BC=\frac{1}{2}$, $AC=\frac{\sqrt{3}}{2}$, $\angle QAT = 90°$, $\angle QCP = 150°$ and RBP is a straight line. Then $[AQT]=\frac{\sqrt{3}}{4}AT=[ART]$ implies $TQ = TR$. Thus



$$\begin{aligned}[PRT]=[PQT]&=\frac{1}{2}[PQR]\\&=\frac{1}{2}\left([ABC]+[ABR]+[BCP]+[ACQ]+[CPQ]-[AQR]\right)\\&=\frac{1}{2}\left(\frac{\sqrt{3}}{8}+\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{16}+\frac{3\sqrt{3}}{16}+\frac{\sqrt{3}}{16}-\frac{\sqrt{3}}{8}\right)\\&=\frac{9\sqrt{3}}{32}\end{aligned}$$

10. Setting $x=y=z=1$, we have $|a+b+c|=1$.

Setting $x=1$ and $y=z=0$, we have $|a|+|b|+|c|=1$.

Setting $(x, y, z) = (1, –1, 0)$, we have $|a-b|+|b-c|+|c-a|=2$.

Since $|a+b+c|=|a|+|b|+|c|$, a, b, c must be of the same sign (unless some of them is/are equal to 0). But

$$2=|a-b|+|b-c|+|c-a|\le(|a|+|b|)+(|b|+|c|)+(|c|+|a|)=2(|a|+|b|+|c|)=2.$$

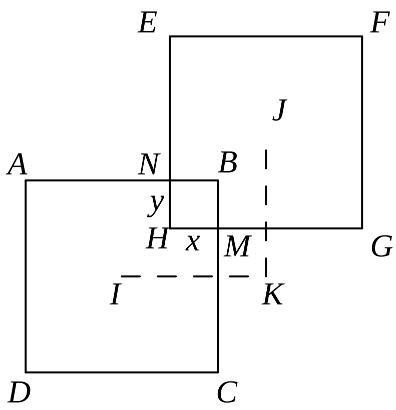
The equality $|a-b|=|a|+|b|$ holds only if a and b are of opposite signs (unless some of them is/are equal to 0). Similarly, b and c have opposite signs, and c and a have opposite signs. Yet a, b, c are of the same sign. Therefore two of a, b, c must be 0, and the other may be 1 or -1. Hence there are 6 possibilities for (a, b, c), namely, $(\pm1, 0, 0)$; $(0, \pm1, 0)$ and $(0, 0, \pm1)$.

11. When n dice are thrown, the smallest possible sum obtained is n and the greatest possible sum obtained is $6n$. The probabilities of obtaining these sums are symmetric about the middle, namely, $\frac{7n}{2}$. In other words, the probabilities of obtaining a sum of S and obtaining a sum of $7n-S$ are the same. (To see this, we may consider the symmetry $1\leftrightarrow 6$, $2\leftrightarrow 5$, $3\leftrightarrow 4$.) Furthermore, the probability for obtaining the different possible sums increases for sums from

n to $\frac{7n}{2}$ and decreases for sums from $\frac{7n}{2}$ to $6n$. Thus the same probability occurs for at most two possible sums. From the above discussions, we know that S is either equal to 2005 or $7n-2005$. Therefore, to minimise S we should minimise n.

Since $2005=6\times 334+1$, n is at least 335. Indeed, when n is 335, S may be equal to $7n-2005=7(335)-2005=340$. Since 340 is smaller than 2005, we conclude that the smallest possible value of S is 340.

12. Let I and J be the centres of $ABCD$ and $EFGH$ respectively, with the orientation as shown. Let AB meet EH at N, BC meet GH at M with $HM=x$ and $HN=y$. Let also K be a point such that $IK\,//\,AB$ and $JK\,//\,EH$. Note that $IK=\frac{1}{2}+\frac{1}{2}-x=1-x$, and similarly $JK=1-y$. Since $xy=\frac{1}{16}$, we have



$$
\begin{aligned}
IJ^2 &= IK^2+JK^2 \\
&= (1-x)^2+(1-y)^2 \\
&= x^2+y^2-2(x+y)+2 \\
&= x^2+y^2+2xy-2xy-2(x+y)+2 \\
&= (x+y)^2-2\left(\frac{1}{16}\right)-2(x+y)+2 \\
&= (x+y)^2-2(x+y)+\frac{15}{8} \\
&= (x+y-1)^2+\frac{7}{8}
\end{aligned}
$$

Hence the minimum value of IJ is $\sqrt{\frac{7}{8}}=\frac{\sqrt{14}}{4}$, which occurs when $x+y=1$ and $xy=\frac{1}{16}$, i.e. $\{x, y\}=\left\{\frac{2+\sqrt{3}}{4}, \frac{2-\sqrt{3}}{4}\right\}$.

13. Suppose the ant starts at (0, 0, 0) and stops at (1, 1, 1). In each step, the ant changes exactly one of the x-, y- or z- coordinate. The change is uniquely determined: either from 0 to 1 or from 1 to 0. Also, the number of times of changing each coordinate is odd. Hence the number of changes in the three coordinates may be 5-1-1 or 3-3-1 (up to permutations). It follows that the answer is $3\times C_5^7+3\times C_3^7\times C_1^4=546$.

14. We first show that student 333 is a 'good student'. Divide the 997 students (all except students 333, 666 and 999) into the following 498 sets:

$$\{1, 2, 4, \ldots\}, \{3, 6, 12, \ldots\}, \{5, 10, 20, \ldots\}, \ldots, \{331, 662\}, \{335, 670\}, \ldots, \{995\}, \{997\}$$

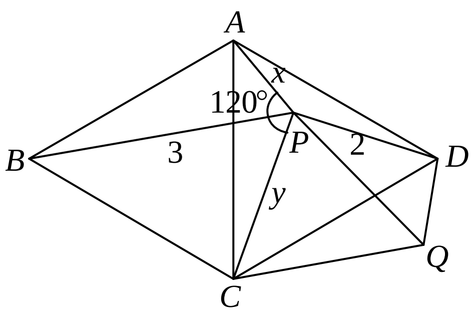
Now consider a group of 500 students containing student 333. If either student 666 or 999 is in the group, then it is a 'good group'. Otherwise, the other 499 students of the group come from the above 498 sets, and the pigeon-hole principle asserts that two of them come from the same set. For any two students coming from the same set, one must have a student number which divides the other's. It follows that this must be a 'good group', and hence student 333 is a 'good student'.

Finally we show that students 334 to 1000 are not 'good students'. Clearly, students 501 to 1000 form a 'bad group', and hence students 501 to 1000 are not 'good students'. Now consider the group formed by the following 500 students:

$$334, 335, \ldots, 667, 669, 671, 673, 997, 999.$$

This group consists of students 334 to 667 and all odd-numbered students from 669 to 999. It is easy to see that these students form a 'bad group', because for $334 \le n \le 500$, student n is in the group but student $2n$ is not, and $3n$ already exceeds 1000. For $n \ge 501$, $2n$ already exceeds 1000. Hence we can't find two students for which the student number of one divides that of another. It follows that this is a 'bad group', and hence students 334 to 500 are not 'good students'. Consequently, the answer is 333.

15. Note that ΔABC and ΔACD are equilateral. Rotate ΔACP clockwise through $60°$ about C to ΔDCQ. Note that ΔPCQ is equilateral and hence APQ is a straight line. Also, we have $\Delta AQC \cong \Delta BPC$. Since $\angle APC + \angle ABC = 180°$, $ABCP$ is a cyclic quadrilateral. Hence $\angle DQP = \angle APB = \angle ACB = 60°$. Let $AP = x$ and $CP = y$. Then



$$3 = BP = AQ = AP + PQ = x + y.$$

On the other hand, applying cosine law in ΔDQP gives $2^2 = x^2 + y^2 - 2xy\cos 60°$. These two equations give $x^2 + y^2 + 2xy = 9$ and $x^2 + y^2 - xy = 4$ respectively. Thus $xy = \frac{5}{3}$ and hence $x^2 + y^2 - 2xy = 4 - \frac{5}{3} = \frac{7}{3}$. Thus $(x - y)^2 = \frac{3}{7}$, or $(x - y) = \sqrt{\frac{7}{3}}$, i.e. the difference between the lengths of AP and CP is $\sqrt{\frac{7}{3}} = \frac{\sqrt{21}}{3}$.

16. Since n leaves a remainder of 502 when divided by 802, we may write $n = 802k + 502$ for some integer k. Let $k = 5h + r$ for some integers h and r such that $0 \le r \le 4$. Then we have

$$n = 802(5h + r) + 502 = 2005(2h) + (802r + 502).$$

Setting r = 0, 1, 2, 3, 4 respectively, we see that the remainders when n is divided by 2005 are 502, 1304, 101, 903 and 1705 respectively. Finally, it is indeed possible for the remainder to be 101. The reason is as follows. According to the question, we know that $n + 300$ is divisible by both 902 and 702, and hence by their L.C.M. (say L). Since L and 2005 are relatively prime, the Chinese remainder theorem asserts that there exists a positive integer n which is congruent to –300 modulo L and to 101 modulo 2005. It follows that the answer is 101.

17. Let a_n denote the number held by children n. Since the sum of the numbers of any 2005 consecutive children is equal to 2005, we have $a_n = a_{n+2005}$ for all n, where the index is taken modulo 5555. In particular, $a_0 = a_{2005k}$ for all positive integers k. Since $(2005, 5555) = 5$, there exists a positive integer k such that $2005k \equiv 5$ (mod 5555). Hence $a_0 = a_5$, and in general $a_n = a_{n+5}$ for all positive integers n. Therefore the sum of the numbers of any 5 consecutive children is the same, and should be equal to $2005 \times \frac{5}{2005} = 5$. In particular, since {1, 12, 123, 1234, 5555} form a complete residue system modulo 5, we must have $a_1 + a_{12} + a_{123} + a_{1234} + a_{5555} = 5$, and hence the answer is $5 - 1 - 21 - 321 - 4321 = -4659$.

18. Note that

$$r - \frac{1}{r} = \frac{a(b-c)}{b(c-a)} - \frac{c(b-a)}{b(c-a)} = \frac{ab - ac - cb + ca}{b(c-a)} = \frac{b(a-c)}{b(c-a)} = -1.$$

Hence $r + 1 - \frac{1}{r} = 0$, i.e. $r^2 + r - 1 = 0$. Solving, we get $r = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$. Since $r > 0$, we take the positive root to get $r = \frac{\sqrt{5} - 1}{2}$.

19. Note that $B_n = (l_1 + l_2 + l_3 + \cdots + l_n, 0)$. Set $S_n = l_1 + l_2 + \cdots + l_n$. The equation of $B_{n-1}A_n$ is $y = \sqrt{3}(x - S_{n-1})$. Hence the y-coordinate of A_n satisfies

$$y_n = \sqrt{3}(y_n{}^2 - S_{n-1}).$$

Solving under the condition $y_n > 0$, we get

$$y_n = \frac{1 + \sqrt{1 + 12S_{n-1}}}{2\sqrt{3}}.$$

Since $l_n = \frac{y_n}{\sin 60^\circ}$, we have

$$\frac{\sqrt{3}}{2} l_n = \frac{1+\sqrt{1+12S_{n-1}}}{2\sqrt{3}},$$

i.e. $(3l_n - 1)^2 = 1 + 12S_{n-1}$. Upon simplification, we have

$$3l_n^2 - 2l_n = 4S_{n-1} \qquad \ldots(1)$$

and hence

$$3l_{n+1}^2 - 2l_{n+1} = 4S_n \qquad \ldots(2)$$

$(2)-(1)$:

$$3(l_{n+1}^2 - l_n^2) - 2(l_{n+1} - l_n) = 4(S_n - S_{n-1})$$
$$3(l_{n+1} + l_n)(l_{n+1} - l_n) - 2(l_{n+1} - l_n) = 4l_n$$
$$3(l_{n+1} + l_n)(l_{n+1} - l_n) - 2(l_{n+1} + l_n) = 0$$

Since $l_n \neq l_{n+1}$, we have

$$l_{n+1} - l_n = \frac{2}{3}$$

Thus $\{l_n\}$ is an arithmetic sequence with common difference $\frac{2}{3}$. Furthermore,

$$\frac{\sqrt{3}}{2} l_1 = \frac{1+\sqrt{1+12S_0}}{2\sqrt{3}},$$

where S_0 is taken to be zero. This gives $l_1 = \frac{2}{3}$. It follows that

$$l_1 + l_2 + \cdots + l_{2005} = \frac{2}{3}(1+2+\cdots 2005) = \frac{2}{3}\cdot\frac{2005\cdot 2006}{2} = \frac{4022030}{3}.$$

Remark. The condition that B_1, B_2, … are distinct is missing in the original question. This is necessary to ensure that $l_n \neq l_{n+1}$ so that we can cancel out the term $l_{n+1} - l_n$.

20. Applying the product-to-sum formula, we have

$$\sin B = 2005\cos(A+B)\sin A$$
$$= \frac{2005}{2}\left[\sin(2A+B)\sin(-B)\right]$$
$$\frac{2007}{2}\sin B = \frac{2005}{2}\sin(2A+B)$$
$$\sin B = \frac{2005}{2007}\sin(2A+B)$$

Hence $\sin B \leq \frac{2005}{2007}$, and so $\tan B \leq \frac{2005}{\sqrt{2007^2 - 2005^2}} = \frac{2005}{2\sqrt{2006}} = \frac{2005\sqrt{2006}}{4012}$. Equality is possible when $\sin(2A+B) = 1$, i.e. when $2A+B$ is a right angle.