

International Mathematical Olympiad
國際數學奧林匹克
Preliminary Selection Contest 2002 – Hong Kong
2002 香港選拔賽

26 May 2002

Answer ALL questions
本卷各題全答

Put your answers on the answer sheet
請將答案寫在答題紙上

Using calculator is NOT allowed
不可使用計算機

Time allowed: 3 hours
時限：3 小時

1. (1 mark) Let n be a positive integer such that no matter how 10^n is expressed as the product of two positive integers, at least one of these two integers contains the digit 0. Find the smallest possible value of n .
(1 分) 設 n 是滿足以下條件的正整數：無論怎樣將 10^n 分解為兩個正整數的積，其中一個數必包含“0”這數字。求 n 的最小可能值。
2. (1 mark) A clock has an hour hand of length 3 and a minute hand of length 4. From 1:00 am to 1:00 pm of the same day, find the number of occurrences when the distance between the tips of the two hands is an integer.
(1 分) 一時鐘的時針長為 3，分針長為 4。問在同一天上午一時至下午一時的一段時間內，時針與分針的端點的距離為整數多少次？
3. (1 mark) Find the sum of all integers from 1 to 1000 which contain at least one “7” in their digits.
(1 分) 求由 1 至 1000 內所有包含 “7” 這數字的整數的總和。
4. (1 mark) A multiple choice test consists of 100 questions. If a student answers a question correctly, he will get 4 marks; if he answers a question wrongly, he will get -1 mark. He will get 0 mark for an unanswered question. Determine the number of different total marks of the test. (A total mark can be negative.)
(1 分) 選擇題測驗共有 100 條問題。每題答對得 4 分，不作答得 0 分，答錯得 -1 分。問這測驗可以有多少個不同的總分？(可以有負分。)
5. (1 mark) A positive integer is said to be a “palindrome” if it reads the same from left to right as from right to left. For example 2002 is a palindrome. Find the sum of all 4-digit palindromes.
(1 分) 一個正整數如果由左邊或右邊讀起都一樣，則稱為「回文數」，例如 2002 是一個四位回文數。求所有四位回文數的總和。
6. (1 mark) Points A and B lie on a plane. A straight line passing through A will divide the plane into 2 regions. A further straight line through B will altogether divide the plane into 4 regions, and so on. If 1002 and 1000 straight lines are drawn passing through A and B respectively, what is the maximum number of regions formed?
(1 分) A 和 B 是平面上的兩個點。經過 A 點的一條直線將平面分為兩部分，經過 B 點的另一條直線再將平面共分為四部分。現各有 1002 及 1000 條直線，分別經過 A 點和 B 點，問這些直線最多可將平面分為多少部分？

7. (1 mark) In $\triangle ABC$, X, Y are points on BC such that $BX = XY = YC$, M, N are points on AC such that $AM = MN = NC$. BM and CN intersect AY at S and R respectively. If the area of $\triangle ABC$ is 1, find the area of $SMNR$.
(1 分) 在 $\triangle ABC$ 中, X, Y 為 BC 上的點使得 $BX = XY = YC$, M, N 為 AC 上的點使得 $AM = MN = NC$. BM 和 CN 分別交 AY 於 S 和 R . 若 $\triangle ABC$ 的面積為 1, 求 $SMNR$ 的面積。
8. (1 mark) Given that $0.3010 < \log 2 < 0.3011$ and $0.4771 < \log 3 < 0.4772$. Find the leftmost digit of 12^{37} .
(1 分) 設 $0.3010 < \log 2 < 0.3011$ 及 $0.4771 < \log 3 < 0.4772$, 求 12^{37} 最左的一位數字。
9. (1 mark) Let x_1, y_1, x_2, y_2 be real numbers satisfying the equations $x_1^2 + 5x_2^2 = 10$, $x_2y_1 - x_1y_2 = 5$ and $x_1y_1 + 5x_2y_2 = \sqrt{105}$. Find the value of $y_1^2 + 5y_2^2$.
(1 分) 設 x_1, y_1, x_2, y_2 為實數, 且 $x_1^2 + 5x_2^2 = 10$, $x_2y_1 - x_1y_2 = 5$ 和 $x_1y_1 + 5x_2y_2 = \sqrt{105}$. 求 $y_1^2 + 5y_2^2$ 的值。
10. (1 mark) How many positive integers less than 500 have exactly 15 positive integer factors?
(1 分) 小於 500 的正整數中有多少個剛好有 15 個正因子?
11. (1 mark) Find the 2002nd positive integer that is not the difference of two square integers.
(1 分) 求第 2002 個不能寫成兩個平方整數的差的正整數。
12. (1 mark) In trapezium $ABCD$, $BC \perp AB, BC \perp CD$ and $AC \perp BD$. Given $AB = \sqrt{11}$ and $AD = \sqrt{1001}$. Find BC .
(1 分) 在梯形 $ABCD$ 中, $BC \perp AB, BC \perp CD$ 及 $AC \perp BD$. 若 $AB = \sqrt{11}$, $AD = \sqrt{1001}$, 求 BC .
13. (2 marks) Let $ABCD$ be a square of side 5, E a point on BC such that $BE = 3, EC = 2$. Let P be a variable point on the diagonal BD . Determine the length of BP if $PE + PC$ is smallest.
(2 分) 正方形 $ABCD$ 邊長為 5, E 為 BC 上的點使得 $BE = 3, EC = 2$. 若 P 是對角線 BD 上的點使 $PE + PC$ 的值最小, 求 BP 的長度。
14. (2 marks) In $\triangle ABC$, $\angle ACB = 3\angle BAC$, $BC = 5, AB = 11$. Find AC .
(2 分) 在 $\triangle ABC$ 中 $\angle ACB = 3\angle BAC$, $BC = 5, AB = 11$. 求 AC .
15. (2 marks) In $\triangle ABC$, D, E and F are respectively the midpoints of AB, BC and CA . Furthermore $AB = 10, CD = 9, CD \perp AE$. Find BF .
(2 分) 在 $\triangle ABC$ 中, D, E 和 F 分別為 AB, BC 和 CA 的中點。且 $AB = 10, CD = 9, CD \perp AE$. 求 BF 。

16. (2 marks) Each face and each vertex of a regular tetrahedron is coloured red or blue. How many different ways of colouring are there? (Two tetrahedrons are said to have the same colouring if we can rotate them suitably so that corresponding faces and vertices are of the same colour.

(2分) 在正四面體的每一個頂點和每一個面都塗上紅色或藍色，問有多少種不同的塗色方法？(若能將兩個已塗色的四面體適當旋轉使得對應的頂點和面同色，則這兩個四面體只能算為一種塗色方法。)

17. (2 marks) Let $a_0 = 2$ and for $n \geq 1$, $a_n = \frac{\sqrt{3}a_{n-1} + 1}{\sqrt{3} - a_{n-1}}$. Find the value of a_{2002} in the form $p + q\sqrt{3}$ where p and q are rational numbers.

(2分) 設 $a_0 = 2$ ，且對 $n \geq 1$ 時 $a_n = \frac{\sqrt{3}a_{n-1} + 1}{\sqrt{3} - a_{n-1}}$ 。用 $p + q\sqrt{3}$ 的形式表示 a_{2002} 的值 (其中 p 和 q 為有理數)。

18. (2 marks) Let $A_1A_2 \cdots A_{2002}$ be a regular 2002-sided polygon. Each vertex A_i is associated with a positive integer a_i such that the following condition is satisfied: If j_1, j_2, \dots, j_k are positive integers such that $k < 500$ and $A_{j_1}A_{j_2} \cdots A_{j_k}$ is a regular k -sided polygon, then the values of $a_{j_1}, a_{j_2}, \dots, a_{j_k}$ are all different. Find the smallest possible value of $a_1 + a_2 + \cdots + a_{2002}$.

(2分) 設 $A_1A_2 \cdots A_{2002}$ 為正 2002 邊形。每一頂點 A_i 對應正整數 a_i 使下列條件成立：如果 j_1, j_2, \dots, j_k 為正整數， $k < 500$ ，而 $A_{j_1}A_{j_2} \cdots A_{j_k}$ 為正 k 邊形時，則 $a_{j_1}, a_{j_2}, \dots, a_{j_k}$ 的值互不相同。求 $a_1 + a_2 + \cdots + a_{2002}$ 的最小可能值。

19. (3 marks) There are 5 points on the plane. The following steps are used to construct lines. In step 1, connect all possible pairs of the points; it is found that no two lines are parallel, nor any two lines perpendicular to each other, also no three lines are concurrent. In step 2, perpendicular lines are drawn from each of the five given points to straight lines connecting any two of the other four points. What is the maximum number of points of intersection formed by the lines drawn in step 2, including the 5 given points?

(3分) 在平面上有 5 點。現用以下步驟畫上直線。

步驟 1：將任何 2 點連接起來。

直線畫好之後，發覺沒有兩條線是平行的，沒有兩條線是互相垂直的，也沒有三條線交於一點。

步驟 2：由原先的 5 點任何一點出發，畫直線使垂直於其餘四點間任二點的連線。

問在步驟 2 的過程所形成的線之間最多共有多少個交點(包括原先的 5 點)？

20. (3 marks) A rectangular piece of paper has integer side lengths. The paper is folded so that a pair of diagonally opposite vertices coincide, and it is found that the crease is of length 65. Find a possible value of the perimeter of the paper.

(3分) 一長方形紙張的邊長為整數。將紙張對摺使得其中兩個對角重疊，得摺線的長度為 65。求該長方形紙張週長的一個可能值。

End of Paper
全卷完