International Mathematical Olympiad 2003
HK Preliminary Selection Contest (May 26, 2002)

Solutions

1. We only need to consider the case $10^n = 2^n \times 5^n$. (In other cases, there will be a zero as unit digit for one of the two factors.) For $n \le 7$, 2^n and 5^n do not contain the digit 0. For $n = 8$, $5^8 = 390625$. So answer is $\underline{\underline{8}}$.

2. Between 1:00 am and 1:00 pm, the hands overlap 11 times. Between two overlappings there are 11 times when the distance is an integer (2 times for distances equal 2, 3, 4, 5, 6, and 1 time for distance equals 7). After the last overlapping (12:00 noon), and before 1:00 pm, again 11 times when distances are integers. Thus altogether $11 + 11 \times 10 + 11 = \underline{\underline{132}}$ times that distances are integers.

3. The sum of the numbers from 700 to 799 is $\frac{1}{2} \times 799 \times 800 - \frac{1}{2} \times 699 \times 700 = 74950$. The sum of the numbers from 70 to 79 is $\frac{1}{2} \times 79 \times 80 - \frac{1}{2} \times 69 \times 70 = 745$. Hence all numbers that end from 70 to 79 (excluding those starting from 7, as we have already counted those from 700 to 799) is $745 \times 9 + 10\left(100 + 200 + \cdots + 600 + 800 + 900\right) = 44705$. The sum of all numbers ending in 7, (excluding the previous two cases), is $9\left(7 + 17 + \cdots + 67 + 87 + 97\right) + 9\left(100 + 200 + \cdots + 600 + 800 + 900\right) = 38187$. So the total sum of numbers containing a " 7 " is $74950 + 47705 + 38187 = \underline{\underline{157842}}$.

Alternate Solution

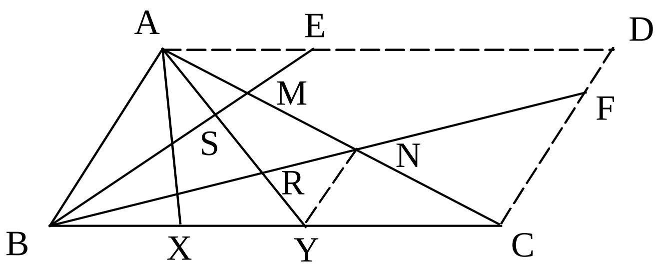
Let P and Q be respectively the sum of all positive integers less than 1000 and the sum of all positive integers containing no "7"s in their digits. Clearly $P = \frac{999 \times 1000}{2} = 499500$. On the other hand, we can easily see Q is the sum of $9^3 = 729$ numbers. Considering the unit digits of these numbers, there should be $81 (= \frac{729}{9})$ occurrences for each digit except 7. Therefore, the sum of all unit digits equals $81 \times \left(0 + 1 + 2 + \ldots\ldots + 6 + 8 + 9\right) = 81 \times 83$. The same is true for the tenth and hundredth digits. Hence, $Q = 81 \times 83 \times \left(1 + 10 + 100\right) = 341658$. Finally, the answer required is therefore $P - Q = \underline{\underline{157842}}$

4. If a student answers 100 questions correctly, he gets 400 marks. If he answers 99 questions correctly, depending if he answers or does not answer the remaining question, he gets 396 or 395 marks. If a student answers 98 questions correctly, depending if he answers or does not answer the remaining 2 questions, he gets 390, 391, or 392 marks. Hence the total marks cannot be 389, 393, 394, 397, 398, 399. If a student answers 97 (or fewer) questions correctly, he get at most 388 marks, by adjusting the questions he answers wrongly, he gets everything possible " in between ". Thus the number of different " total marks " $= 100 + 401 - 6 = \underline{\underline{495}}$.

5. Each 4-digit palindrome is of the form $\overline{abba}$, where a ranges from 1 to 9 and b from 0 to 9. Hence the sum of all 4-digit palindromes is equal to $\sum_{n=0}^{9}\sum_{m=1}^{9}(1001m + 110n)$
$= \sum_{n=0}^{9}(1001\times 45 + 110n\times 9) = 1001\times 45\times 10 + 45\times 9\times 110 = 11000\times 45 = \underline{\underline{495000}}$.

6. The 1000 lines through B divide the plane into 2000 regions. For the first line drawn through A, the number of new regions formed is 1001. For each new line drawn through A, the number of new regions formed is 1002. Therefore the total number of regions formed is $1002\times 1002 - 1 + 2000 = \underline{\underline{1006003}}$.

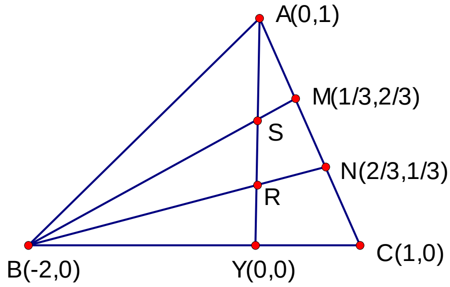
7. Denote by $[BMN]$ the area of ΔBMN, then $[BMN] = \frac{1}{3}$. Join YN, then $YN // BA$ and $\frac{RN}{BR} = \frac{YN}{AB} = \frac{CY}{CB} = \frac{1}{3}$, hence $\frac{BR}{BN} = \frac{3}{4}$. Draw AD and CD such that $AD // BC$, $CD // AB$; extend BM and BN to meet AD and CD at E and F respectively. We have $\frac{AE}{BC} = \frac{AM}{MC} = \frac{1}{2}$, so E is the midpoint of AD and



$BM = \frac{2}{3}BE$. Also $\frac{BS}{SE} = \frac{BY}{AE} = \frac{\frac{2}{3}}{\frac{1}{2}} = \frac{4}{3}$ and so $BS = \frac{4}{7}BE$. Finally we have $\frac{BS}{BM} = \frac{\frac{4}{7}}{\frac{2}{3}} = \frac{6}{7}$. Now $\frac{[BSR]}{[BMN]} = \frac{BS\times BR}{BM\times BN} = \frac{6}{7}\times\frac{3}{4} = \frac{9}{14}$. Therefore $[BSR] = \frac{9}{14}\times\frac{1}{3}[ABC] = \frac{3}{14}$, and $[MNRS] = \left(\frac{1}{3} - \frac{3}{4}\right)[ABC] = \frac{5}{42}[ABC] = \underline{\underline{\frac{5}{42}}}$.

Alternate Solution

The problem is a matter of ratios of areas, and we may take any triangle satisfying the prescribed conditions. Suppose we take $A(0, 1)$, $B(-2,0)$, $C(1,0)$, then we have $Y(0,0)$, $M\left(\frac{1}{3},\frac{2}{3}\right)$ and $N\left(\frac{2}{3},\frac{1}{3}\right)$. AY and BM meet at $S\left(0,\frac{4}{7}\right)$; also AY and BN meet at $R\left(0,\frac{1}{4}\right)$. The area of the



quadrilateral $SRNM$ is $\frac{1}{2}\begin{vmatrix} 0 & \frac{4}{7} \\ 0 & \frac{1}{4} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{4}{7} \end{vmatrix} = \frac{1}{2}\left(\frac{4}{9}+\frac{4}{21}-\frac{1}{6}-\frac{1}{9}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{126}\right)(45)=\frac{15}{84}$. Hence, the ratio

of the areas of the quadrilateral $SRNM$ over the triangle is $\frac{15}{84}/\frac{3}{2}=\underline{\underline{\frac{5}{42}}}$

Remark: The problem may also be solved using the Menelaus' theorem.

8. Check that $\log 12 = 2\log 2 + \log 3$, so we have $1.0791 = 2\times 0.3010 + 0.4771 < \log 12 < 2\times 0.3011 + 0.4772 = 1.0794$, which implies $39 + \log 8 = 39 + 3\log 2 < 39.9267 < 37\log 12 < 39.9378 < 39 + 2\log 3 = 39 + \log 9$. Hence the leftmost digit of 12^{37} is $\underline{\underline{8}}$.

9. $(x_1^2+5x_2^2)(y_1^2+5y_2^2) = (x_1y_1+5x_2y_2)^2 + 5(x_2y_1 - x_1y_2)^2$, thus $10(y_1^2+5y_2^2) = 105 + 5\times 5^2 = 230$, or

$y_1^2 + 5y_2^2 = \underline{\underline{23}}$.

Alternate Solution

From $x_1{}^2 + 5x_2{}^2 = 10$, we may have let $x_1 = \sqrt{10}\cos\alpha$, $x_2 = \sqrt{2}\sin\alpha$. Similarly if we let

$y_1 = r\cos\beta$ and $y_2 = \frac{r}{\sqrt{5}}\sin\beta$, then $y_1{}^2 + 5y_2{}^2 = r^2$. From $x_2 y_1 - x_1 y_2 = 5$ and

$x_1 y_1 + 5x_2 y_2 = \sqrt{105}$, we have

$$\begin{cases} \sqrt{2}\sin\alpha \cdot r\cos\beta - \sqrt{10}\cos\alpha \cdot \frac{r}{\sqrt{5}}\sin\beta = \sqrt{2}r\sin(\alpha+\beta) = 5 \\ \sqrt{10}\cos\alpha \cdot r\cos\beta + 5\sqrt{2}\sin\alpha \cdot \frac{r}{\sqrt{5}}\sin\beta = \sqrt{10}r\cos(\alpha-\beta) = \sqrt{105} \end{cases}.$$

This implies $r^2 = \left(\frac{5}{\sqrt{2}}\right)^2 + \left(\frac{\sqrt{105}}{\sqrt{10}}\right)^2 = \underline{\underline{23}}$

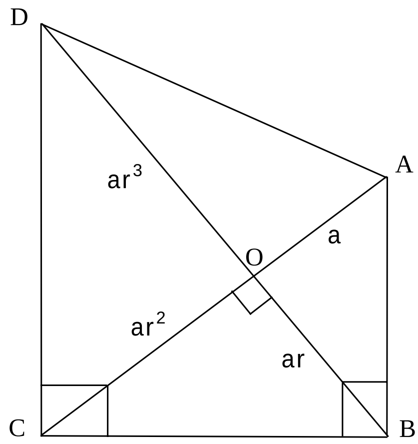
Alternate Solution 2

Apparently any special case should work. Try $x_1 = \sqrt{10}, x_2 = 0$. From $x_2 y_1 - x_1 y_2 = 5$, get $y_2 = -\frac{5}{\sqrt{10}}$. From $x_1 y_1 + 5x_2 y_2 = \sqrt{105}$, get $y_1 = \frac{\sqrt{105}}{\sqrt{10}}$. Hence $y_1^2 + 5y_2^2 = \frac{105}{10} + \frac{5 \cdot 25}{10} = \underline{\underline{23}}$.

10. Since $15 = 1 \times 15 = 3 \times 15$ are the only factorizations of 15, so a positive integer has exactly 15 positive integer factors if and only if it is of the form p^{14} for some prime p, or $p^2 q^4$ for two distinct primes p and q. Now $2^{14} > 500$, so there are no numbers of the first form between 1 and 500. The only one of the second form are $3^2 \times 2^4 = 144$, $5^2 \times 2^4 = 400$, and $2^2 \times 3^4 = 324$, hence $\underline{\underline{3}}$ is the answer.

11. Note that $(n+1)^2 - n^2 = 2n+1$, hence every odd positive integer is the difference of squares of integers. Also $(n+2)^2 - n^2 = 4n + 4 = 4(n+1)$, hence any integer divisible by 4 is the difference of squares of integers. Now for a number of the form $4n+2$, suppose $4n+2 = x^2 - y^2$, then x and y must be of the same parity, hence $x^2 - y^2$ is divisible by 4, while $4n+2$ is not. So the positive integers that cannot be expressed as the difference of two square integers are exactly those of the form $4n+2$. The 2002^{th} member is $4(2001) + 2 = \underline{\underline{8006}}$.

12. Let AC and BD intersect at O. From ΔABC, get $\frac{OA}{OB}=\frac{OB}{OC}$, from ΔBCD, get $\frac{OB}{OC}=\frac{OC}{OD}$. If $OA=a$, $OB=ar$, then $OC=ar^2$, $OD=ar^3$. By Pythagorean theorem, $AB=a\sqrt{1+r^2}$, $AD=a\sqrt{1+r^6}$.



Then $91=\frac{1001}{11}=\frac{AD^2}{AB^2}=\frac{1+r^6}{1+r^2}=r^4-r^2+1.$

Hence $r^4-r^2-90=\left(r^2-10\right)\left(r^2+9\right)=0$, or $r^2=10$, $r=10$.

Since $\sqrt{11}=AB=a\sqrt{1+r^2}$, get $a=1$, and

$BC=\sqrt{OB^2+OC^2}=\underline{\underline{\sqrt{110}}}$

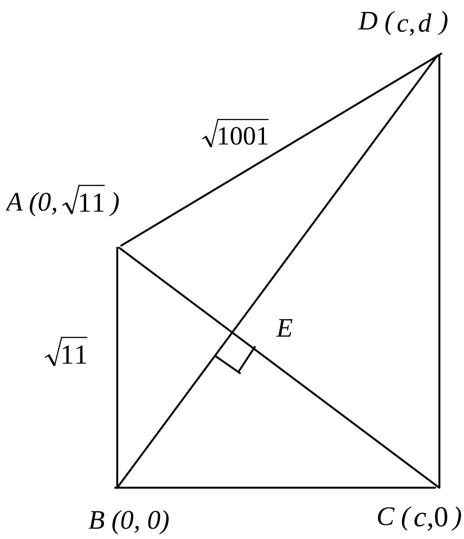
Alternate Solution

Assign the coordinates $A(0,\sqrt{11}), B(0,0), C(1,0)$ and $D(c,d)$.

As $AC\perp BD$, we have $\left(\frac{d}{c}\right)\left(-\frac{\sqrt{11}}{c}\right)=-1$, or $c^2=\sqrt{11}d$.

Hence, $AD^2=c^2+(d-\sqrt{11})^2$

$=c^2+(\frac{c^2}{\sqrt{11}}-\sqrt{11})^2=1001.$



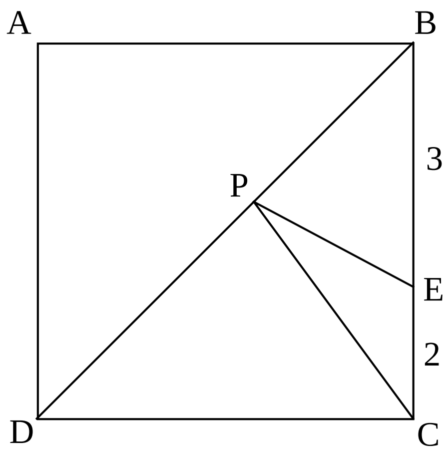
Simplify to get $c^4-11c^2-10890=0.$

Let $c^2=s$, and observe that $10890=99\times 110$,

have $(s-110)(s+99)=0$, or $s=110$,

implying $\underline{\underline{c=\sqrt{110}.}}$.

13. Apply cosine rule to ΔBCP, we have $\cos\angle PEB = \dfrac{BE^2 + PE^2 - PB^2}{2BE \cdot PE}$, $\cos\angle PEC = \dfrac{PE^2 + EC^2 - PC^2}{2PE \cdot EC}$. Using the fact $\cos\angle PEB = -\cos\angle PEC$, and after simplifications, get $PE^2 = PB^2 \cdot \dfrac{EC}{BC} + PC^2 \cdot \dfrac{BE}{BC} - EC \cdot BE$, or $PE^2 = \dfrac{2}{5}PB^2 + \dfrac{3}{5}PC^2 - 6 \ldots$(1) Apply cosine rule to



ΔBCD, we have $\cos\angle CPB = \dfrac{BP^2 + PC^2 - BC^2}{2BP \cdot PC}$, $\cos\angle CPD = \dfrac{DP^2 + PC^2 - CD^2}{2DP \cdot PC}$. Again by $\cos\angle CPB = -\cos\angle CPD$, $BC = CD = 5$, $BP = 5\sqrt{2}$, after simplifications, get $PC^2 = BC^2 - BP \cdot DP = BC^2 - BP(BD - BP) = 25 - BP(5\sqrt{2} - BP) \ldots$(2). Put (2) in (1), get $PE^2 = PB^2 - 3\sqrt{2}BP + 9 \ldots$ (3). Hence $PE + PC = \sqrt{PB^2 - 3\sqrt{2}BP + 9} + \sqrt{PB^2 - 5\sqrt{2}BP + 25}$ $= \sqrt{\left(PB - \dfrac{3}{2}\sqrt{2}\right)^2 + \dfrac{9}{2}} + \sqrt{\left(PB - \dfrac{5\sqrt{2}}{2}\right)^2 + \dfrac{25}{2}}$. Take $PB = x$, then $PE + PC$ corresponds to the sum of distances from (x, 0) to $\left(\dfrac{3}{2}\sqrt{2}, \dfrac{3}{2}\sqrt{2}\right)$, and $\left(\dfrac{5\sqrt{2}}{2}, \dfrac{-5\sqrt{2}}{2}\right)$. For $PE + PC$ to be smallest (x, 0) lies on the line formed by these two points. Hence $\dfrac{\dfrac{3}{2}\sqrt{2} - \left(-\dfrac{5\sqrt{2}}{2}\right)}{\dfrac{3}{2}\sqrt{2} - \dfrac{5\sqrt{2}}{2}} = \dfrac{\dfrac{5\sqrt{2}}{2}}{x - \dfrac{5\sqrt{2}}{2}}$, solve to get

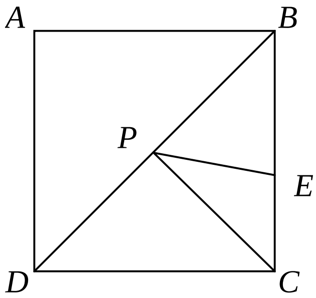
$x = \underline{\underline{\dfrac{15\sqrt{2}}{8}}}$. (Corresponds to $PE + PC = \sqrt{34}$).

Alternate Solution

Clearly $PC = PA$, hence $PE + PC$ is smallest when $PE + PA$ is smallest, i.e. when A, P, E are collinear.

Assign the coordinates $A(0,5)$, $B(5,5)$, $C(5,0)$, $D(0,0)$,

then $E = (5, 2)$, and the equation of the line AE is $y = -\frac{3}{5}x + 5$.

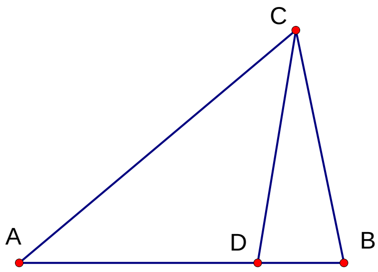


This means $P = \left(\frac{25}{8}, \frac{25}{8}\right)$ and $PB^2 = \left(5 - \frac{25}{8}\right)^2 + \left(5 - \frac{25}{8}\right)^2$

$= \frac{450}{64}$ or $PB = \underline{\underline{\frac{15}{8}\sqrt{2}}}$

14. From C construct $\angle ACD = \angle BAC$, with D on AB. Then $\angle BDC = \angle BCD = 2\angle DAC$. Hence $BC = BD = 5$, $AD = CD = 6$. Apply cosine rule to ΔBDC, get $\angle DBC = \frac{5^2 + 5^2 - 6^2}{2 \times 5 \times 5} = \frac{14}{50}$. Apply cosine rule to ΔABC, get

$AC^2 = 11^2 + 5^2 - 2(5)(11)\cos\angle B = 146 - \frac{154}{5} = \frac{576}{5}$, Hence $AC = \frac{24}{\sqrt{5}} = \underline{\underline{\frac{24\sqrt{5}}{5}}}$.



Alternate Solution

Let $\angle CAB = x < 90^\circ$. By the sine rule, we have $\frac{11}{\sin 3x} = \frac{5}{\sin x} = \frac{AC}{\sin(180^0 - x)}$.

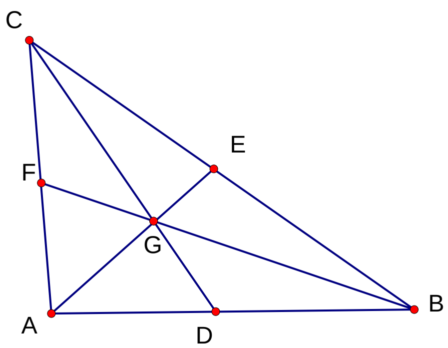
From the first equality, we have $\frac{11}{5} = \frac{\sin 3x}{\sin x} = 3 - 4\sin^2 x$, hence $\sin x = \frac{1}{\sqrt{5}}$, and $\cos x = \frac{2}{\sqrt{5}}$.

From the second equality, we have $AC = \frac{5\sin 4x}{\sin x} = 5(4\cos x(1 - 2\sin^2 x))$

$= (5)(4)\left(\frac{2}{\sqrt{5}}\right)\left(1 - \frac{2}{5}\right) = \frac{24}{\sqrt{5}} = \underline{\underline{\frac{24\sqrt{5}}{5}}}$

(The compound angle formula $\sin(A + B) = \sin A\cos B + \cos A\sin B$ is used repeatedly here.)

15. As shown in the figure, $AB = 10$, $BC = 2x$ and $CA = 2y$. D, E and F are midpoints of AB, BC and CA respectively, G is the centroid, and further $CD \perp AE$, we have $CD = 9$, from the property of centroid $\frac{CG}{GD} = \frac{AG}{GE} = \frac{BG}{GF} = 2$, hence $CG = 6$, $GD = 3$. Let $GE = a$, then $AG = 2a$. By Pythagorean theorem, we have $a^2 + 36 = x^2$, $(2a)^2 + 36 = (2y)^2$, $9 + (2a)^2 = 25$. If follows that $a = 2$, $x = 2\sqrt{10}$ and $y = \sqrt{13}$. Let $\theta = \angle BCA$. Apply cosine rule to ΔBCA, get $100 = (2y)^2 + (2x)^2 - 2(2x)(2y)\cos\theta$, so $xy\cos\theta = 14$. Apply cosine rule to ΔBCF, get $BF^2 = y^2 + (2x)^2 - 2y(2x)\cos\theta = 13 + 160 - 4 \times 14 = 117$. Hence $BF = \sqrt{117} = \underline{\underline{3\sqrt{13}}}$.



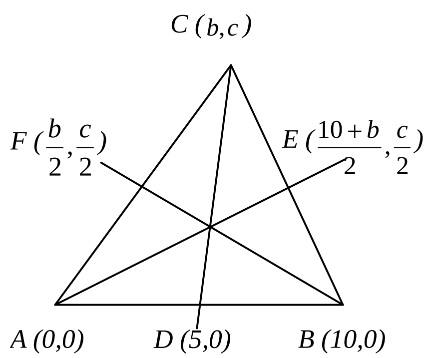
__Alternate Solution__

Choose the coordinates of A, B and C as (0,0), (10,0) and (b, c) respectively. Then the coordinates of D, E and F are $(5,0)$, $(\frac{10+b}{2}, \frac{c}{2})$ and $(\frac{b}{2}, \frac{c}{2})$ respectively.

As $CD = 9$, hence $(b-5)^2 + c^2 = 81$, or $b^2 + c^2 = 10b + 56$.......(1)

Also $CD \perp AE$, get $\frac{c}{10+b} \cdot \frac{c}{b-5} = -1$, or $b^2 + c^2 = -5b + 50$.......(2)

(1) and (2) imply $b = -\frac{2}{5}$.



Finally $BF^2 = \left(\frac{c}{2}\right)^2 + \left(\frac{b}{2} - 10\right)^2 = \frac{1}{4}c^2 + \frac{1}{4}(b-20)^2$

$= \frac{1}{4}(b^2 + c^2) - 10b + 100 = \frac{1}{4}(52) + 4 + 100 = 117$.

Hence $BF = \sqrt{117} = \underline{\underline{3\sqrt{13}}}$

16. Let x be the number of blue vertices and y be the number of blue faces in a coloring. If $x=0$, we have 5 different ways to color the faces, namely $y=0,\ 1,\ 2,\ 3,\ 4$. If $x=1$, there are 2 ways to color the face opposite to the blue vertex and the remaining three faces may have 0, 1, 2 or 3 blue faces. Thus there are $2\times 4=8$ different ways in this case. If $x=2$, call the edge connecting the two blue vertices the " main edge " and the two faces common to this edge the " main faces ". There are 3 ways to color the main faces, namely, both red, both blue, and one red and one blue. In the former two cases there are 3 ways to color the remaining faces, while in the last case there are four because reversing the color of the two faces does matter. Hence there are 10 ways in this case. By symmetry, there are 8 ways corresponding to the case $x=3$ and 5 ways corresponding to $x=4$. Hence the answer is $5+8+10+8+5=\underline{\underline{36}}$.

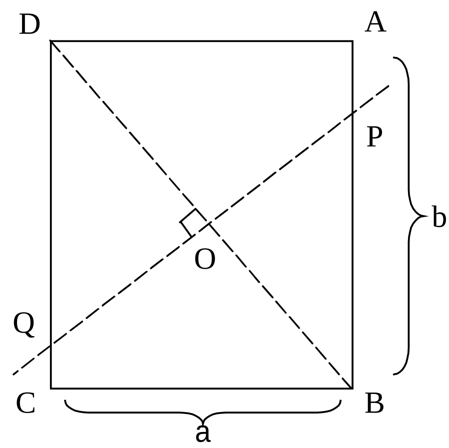
17. Rewrite the recurrence relation as $a_n=\dfrac{a_{n-1}+\dfrac{1}{\sqrt{3}}}{1-\dfrac{1}{\sqrt{3}}a_{n-1}}$. Define $\theta_n=\tan^{-1}a_n$, then the above is equivalent to $\tan\theta_n=\dfrac{\tan\theta_{n-1}+\tan\dfrac{\pi}{6}}{1-\tan\dfrac{\pi}{6}\tan\theta_{n-1}}=\tan\left(\theta_{n-1}+\dfrac{\pi}{6}\right)$. Thus $\theta_n=\theta_{n-1}+\dfrac{\pi}{6}$ ("modulo" π if necessary) and thus $a_{n+6}=\tan\theta_{n+6}=\tan(\theta_n+\pi)=\tan\theta_n=a_n$. If follows that

$$a_{2002}=a_4=\tan\theta_4=\tan\left(\theta_0+\frac{2\pi}{3}\right)=\frac{\tan\theta_0+\tan\dfrac{2\pi}{3}}{1-(\tan\theta_0)\left(\tan\dfrac{2\pi}{3}\right)}=\frac{2+\left(-\sqrt{3}\right)}{1-(2)\left(-\sqrt{2}\right)}=\underline{\underline{\frac{5\sqrt{3}-8}{11}}}.$$

18. Since $2002=2\times 7\times 11\times 13$, we can choose certain vertices among the 2002 vertices to form regular 7-, 11-, 13-, $\cdots$ polygons (where the number of sides runs through divisors of 2002 greater than 2 and less than 1001). Since $2002=7\times 286$, so there are at least 286 different positive integers among the a_i's. Hence the answer is at least $7\times(1+2+\cdots+286)=\dfrac{7\times 286\times 287}{2}=287\times 1001=287287$. Indeed this is possible if we assign the number $\left[\dfrac{i}{7}\right]+1$ to each A_i, for $i=1,2,\ \cdots,\ 2002$.

19. Let the five given points be A, B, C, D and E. The number of straight lines constructed in step 1 is $C_2^5 = 10$. From each of the five pints A, B, C, D and E, a total of $C_2^4 = 6$ perpendicular lines can be drawn. It seems that the number of intersection points should be $C_2^{30} = 435$. However for any straight line, say AB, the three perpendicular lines drawn from C, D and E are parallel, thus will not meet. Hence the number of intersection points should be cut down by $10 \times C_2^3 = 30$. Next for any of the $C_3^5 = 10$ triangle, formed, say ΔABC, there are three perpendicular lines which are concurrent. Hence the number of intersection points should be further cut down by $10 \times \left(C_2^3 - 1\right) = 20$. Finally the points A, B, C, D and E, each represents the intersection of six lines (lines drawn from each of them), hence the number of intersection points should be cut down by $5 \times \left(C_2^6 - 1\right) = 70$. The required number of points of intersection is therefore $435 - 30 - 20 - 70 = \underline{\underline{315}}$.

20. Let $ABCD$ be the piece of paper, with $AB = a \le b = BC$. Let O be the center of the rectangle, B be folded to coincide with D, and POQ be the crease, with P on BC and Q on DA. Then PQ is the perpendicular bisector of BD, and $\Delta BPO \sim \Delta DBC$. Let x be the length of PQ, then



$$\frac{\frac{1}{2}\sqrt{a^2+b^2}}{\frac{1}{2}x} = \frac{b}{a}, \text{ or } x = \frac{a}{b}\sqrt{a^2+b^2}. \text{ Now}$$

$$x = \frac{a}{b}\sqrt{a^2+b^2} = 65, \text{ thus } b = \frac{a^2}{\sqrt{65^2 - a^2}}. \text{ For}$$

b to be an integer, the denominator $\sqrt{65^2 - a^2}$ must be an integer. Hence we try 65^2 as a sum of two squares. Using $5^2 = 3^2 + 4^2$ and $13^2 = 5^2 + 12^2$ we have $65^2 = (13 \times 3)^2 + (13 \times 4)^2 = (5 \times 5)^2 + (12 \times 5)^2 = (3 \times 12 + 4 \times 5)^2 + (4 \times 12 - 3 \times 5)^2 = (3 \times 12 - 4 \times 5)^2 + (4 \times 12 + 3 \times 5)^2$. Consider all these a which may be 16, 25, 33, 39, 52, 56, 60 or 63. Put these to find b, and bear in mind b is an integer with $b \ge a$, we get $a = 60$, and $b = 144$ the only possibility. Thus the perimeter is $2 \times (60 + 144) = \underline{\underline{408}}$.

~ End of Paper ~