Assignment 3
Due 30 November 2002

1. A stock price is modeled by

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where μ is the expected return, σ is the volatility of the stock price, and B_t is a standard Brownian motion. At time t, the stock price S_t is given. Evaluate the expected value and variance of the stock price S_T at future time T.

2. A random number X_t is a martingale if

$$X_t = E_t(X_s), \qquad \text{for} \quad s > t. \tag{1}$$

Which of the following random numbers are martingales?

(a) $B_t + 3B_t^3$,
(b) $B_t^2 + 4B_t^4$,
(c) $B_t + e^{B_t}$,
(d) $5B_t^3 + e^{-\frac{1}{2}t+B_t}$,
(e) $e^{-\frac{1}{2}\sigma^2 t+\sigma B_t}$,

where B_t is a standard Brownian motion. Prove your results by using equation (1).

3. The current price of a stock is 60$. An option trader expects that the stock price will go up to 90$ with 60% probability and will fall down to 40$ with 40% probability in six months. The semi-annually compounded interest rate is 10% per annum. What is the risk-neutral probability for the stock price going up? What is the fair price of an at-the-money call option? What are the trader's expected returns (per annum with semi-annual compounding) on the stock and the call option?

4. In a risk-neutral world, a stock price is modeled by

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

where r is risk-free rate, σ is the volatility of the stock price, and B_t is a standard Brownian motion. Given the stock price S_t, write down the transition probability density function of S_T at future time T. Evaluate a *binary call* option with the following payoff

$$V(S_T, T) = \begin{cases} 1 & S_T \geq K \\ 0 & S_T < K \end{cases},$$

by using Harrison-Pliska's (1981) risk-neutral valuation formula

$$V(S_t, t) = e^{-r(T-t)} E_t^Q[V(S_T, T)],$$

where E_t^Q stands for the conditional expectation under risk-neutral probability measure.

5. The price of a *binary put* option can be obtained by solving Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$
$$V(S,T) = \begin{cases} 0 & S_T \geq K \\ 1 & S_T < K \end{cases}.$$

Find the price formula for the binary put by solving the PDE with transformation, heat equation and Green's function approach. Combining the results of Question 3 and 4, construct a parity relation for a binary put and a binary call with the same strike price K.