Lemma: Cauchy-Schwarz Inequality in the context of expectation.

Let X and Y be two random variables such that $\mathbb{E}(X)$, $\mathbb{E}(Y)$, and $\mathbb{E}(XY)$ exist. Then

$$[\mathbb{E}(XY)]^2 \leq \mathbb{E}(X^2)\mathbb{E}(Y^2)$$

Pf.

Consider a new random variable $X - \alpha Y$.

$$0 \leq \mathbb{E}[(X - \alpha Y)^2] = \mathbb{E}(X^2) - 2\alpha\mathbb{E}(XY) + \alpha^2\mathbb{E}(Y^2)$$

If we now let $\alpha = \frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}$ and plug into the above inequality, we get

$$0 \leq \mathbb{E}(X^2) - \frac{2[\mathbb{E}(XY)]^2}{\mathbb{E}(Y^2)} + \frac{[\mathbb{E}(XY)]^2}{\mathbb{E}(Y^2)}$$

$$\frac{[\mathbb{E}(XY)]^2}{\mathbb{E}(Y^2)} \leq \mathbb{E}(X^2)$$

This is just what we wanted to prove. ◘

Theorem. "Correlation is bounded by 1."

Let X and Y be two random variables such that $\mathbb{E}(X)$, $\mathbb{E}(Y)$, and $\mathbb{E}(XY)$ exist. Then

$$|\mathrm{Corr}(X, Y)| \leq 1$$

Pf.

By the lemma, we have

$$(\mathbb{E}\{[X - \mathbb{E}(X)] \cdot [Y - \mathbb{E}(Y)]\})^2 \leq \underbrace{\mathbb{E}([X - \mathbb{E}(X)]^2)}_{\mathbb{V}(X)} \cdot \underbrace{\mathbb{E}([Y - \mathbb{E}(Y)]^2)}_{\mathbb{V}(Y)}$$

$$\frac{(\mathbb{E}\{[X - \mathbb{E}(X)] \cdot [Y - \mathbb{E}(Y)]\})^2}{\mathbb{V}(X) \cdot \mathbb{V}(Y)} \leq 1$$

This is just what we wanted to prove. ◘