On the oscillatory and mean motions due to waves in a thin viscoelastic layer

Xueyan Zhang, Chiu-On Ng *

Department of Mechanical Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong

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Abstract

A perturbation analysis based on equations of motion in Lagrangian form is presented for the oscillatory and mean motions induced by a forced periodic wave propagating in a thin layer of viscoelastic material. The material is assumed to be a Voigt body, for which the constitutive relation is a linear combination of viscous and elastic parts. In this work, the elastic part of the stress is assumed to be a linear function of the Lagrangian deformation tensor. The aim is to show a proper approach of carrying out the analysis to the second order in order to determine the mean deformation undergone by the material. This approach is in sharp contrast to the previous studies, which have mistakenly applied the complex viscoelastic parameter to the second order and assumed the mean motion to be a steady Lagrangian drift. It is shown here that for a sufficiently soft and viscous material the mean motion is actually a creeping motion that slowly dies out as a limit of finite deformation is approached. Numerical results are also generated to illustrate the combined effects due to viscous damping and elasticity on the first-order oscillatory and the second-order mean displacements of particles in the material.

Keywords: Viscoelastic material; Mass transport in waves; Wave-mud interaction

1. Introduction

Coastal and estuarine sea beds are often loaded with cohesive sediments (also known as marine deposits or mud), the transport of which under the forcing of surface waves and currents has long been a central topic in sediment transport. The problem of cohesive sediment transport is of practical concern, not only because of the transport of the sediment particles themselves, but also because of the concomitant transport of contaminants that may be sorbed on the sediment particles. Cohesive sediments are known to be rich in organic matter and their particles are very tiny but with a large specific area that is ionic in nature. These properties enable the sediments to have a strong potential of sorbing organic chemical and heavy metal pollutants onto themselves. It is common to find aquatic systems in which a substantial amount of the pollutants reside in the benthic deposits. Without a good understanding of the various processes controlling the sediment transport, it

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would be impossible to accurately estimate the fate and transport of pollutants that have reached the bottom of the system.

Surface waves and the bed mud interact with each other such that the mud can enhance the wave attenuation, while the waves can induce a Lagrangian drift [1] on the bottom driving a slow but steady mass transport of the mud. The mechanics of waves over mud has been studied extensively with a focus on these two aspects. One crucial element in modeling the response of mud to waves is a proper description of the rheology of the mud itself. Since the geo-environmental conditions under which natural muds are formed are diverse and site-specific, it is reasonable to expect that mud at different localities and in different periods of time may exhibit vastly different rheological behaviors. It is not uncommon to find marine muds whose rheological curves are non-linear, and may even exhibit hysteresis. Analytical work is however possible only for a simple linear material. This explains why it prevails in the classical literature of wave–mud interaction that the mud is assumed to be either a Newtonian viscous fluid (e.g., [2,3,4]), or a linear elastic solid (e.g., [5,6]), or a poroelastic medium (e.g., [7,8]).

In reality, a muddy bed behaves more like a viscous fluid towards the top layer of fluid mud formed by disturbance of wave action, but more like an elastic solid towards the deeper settled and consolidated bed. For the sediment that is newly deposited and partially consolidated, the medium may possess the combined properties of fluid and solid, and is more appropriately modeled as a viscoelastic material. This has motivated various studies into the interaction between waves and viscoelastic mud, as presented by Hsiao and Shemdin [9], MacPherson [10], Maa and Mehta [11], Shibayama et al. [12], Piedra-Cueva [13], Foda et al. [14], Hill and Foda [15], Soltanpour et al. [16], and so on. By and large, these studies have shown that the wave kinematics and dynamics can be much affected by the viscoelasticity of the bottom. The wave damping is significant despite relatively small bottom motions. Also, resonance may occur in a finite elastic layer, which will amplify the energy dissipation in the bottom boundary layers, thereby further enhancing the wave damping rate by an order of magnitude or more.

There is however one key thing regarding the dynamics of a viscoelastic material that seems to have been mishandled by some previous authors. For viscoelastic mud that can be modeled as a Voigt body [17,18], the stress–strain constitutive relationship under simple shear can be written as

\[ \tau = G\gamma + \rho v\dot{\gamma}, \]

where \( \tau \) is the shear stress, \( \gamma \) is the shear deformation, \( \dot{\gamma} \) is the shear rate, \( G \) is the shear modulus, \( \rho \) is the density, and \( v \) is the kinematic viscosity. If the material undergoes a simple harmonic motion such that \( \gamma \) varies with \( \exp(-i\sigma t) \), where \( i = \sqrt{-1} \) is the complex unit, \( \sigma \) is the frequency of oscillation, and \( t \) is the time, then the shear rate is related to the strain according to \( \dot{\gamma} = -i\sigma\gamma \). It follows that the constitutive relationship can be simplified to

\[ \tau = \rho v_e \dot{\gamma}, \]

where \( v_e \) is the complex viscoelastic parameter given by

\[ v_e = v + iG/\sigma \rho, \]

in which the real part is the kinematic viscosity and the imaginary part is a measure of the elasticity. Therefore by substituting \( v_e \) for \( v \) in the fluid momentum equations, one may perform an analysis for the simple harmonic motion of the viscoelastic material. This is the approach that was used by Hsiao and Shemdin [9], and MacPherson [10], and later followed by others, on the linearized analysis of the motions of a viscoelastic medium in response to a monochromatic small-amplitude wave. It is important to realize that any theory based on (3) is valid only to the first order in wave slope. It is fine if one wishes to examine the first-order wave kinematics and dynamics, including the wave damping rate, in the presence of a viscoelastic bottom. It will be a fundamental mistake, however, if one wishes to apply (3) to a second-order analysis. The reason is obvious. The higher-order motions will no longer be simple harmonic, and the strain is no longer related to the strain rate by the simple relationship as shown above. Apparently, this point has not been well appreciated by some authors (e.g., [19,16]), who have used the viscoelastic parameter (3) to determine the mass transport velocity in a layer of viscoelastic mud under waves. Their works are incorrect on two accounts. First, mass transport in waves is a second-order problem, to which the viscoelastic parameter (3) is not applicable. Second, for a material
2. Basic formulation

Consider a thin horizontal layer of viscoelastic material of depth $h$ subject to a periodic pressure load $P$, applied on its free surface. The layer is assumed to be very long in the longitudinal direction, and very wide in the spanwise direction so that the end effects can be ignored. The applied load is in the form of a long-crest ed progressive wave of wavenumber $k$ and angular frequency $\sigma$ propagating along the free surface of the layer. The material motion in response to the load is a forced wave of the same prescribed wavelength and wave period. There can also be an eigen-value type of free wave, whose wavelength and period are related by a dispersion relation. The free wave, if any, will be rapidly damped by the viscosity [23], and is ignored in this study.

The material is assumed to be homogeneous and isotropic with constant density $\rho$, and viscoelastic parameters of dynamic viscosity $\mu$ and shear modulus $G$. Since small but higher-order displacements of the material are considered in this study, we adopt the Lagrangian description for the analysis. We let $(x, \beta)$ be the undisturbed horizontal and vertical coordinates of a material particle, respectively, and $(\xi, \eta)$ the coordinates of the particle at time $t > 0$. The material motions are to be found in terms of $\xi, \beta$, and $t$.

In this study, we follow the same assumptions as those made by Ng [21]. The core assumptions are shallowness and small disturbances. First, the boundary-layer approximations are to be applied. The wave motion of the material is to result in an oscillatory (or Stokes) boundary layer of thickness $\delta = (2\nu/\alpha)^{1/2}$ (where $\nu = \mu/\rho$ is the kinematic viscosity of the material), which is assumed to be comparable with the material layer thickness $h$, but is much shorter than the wavelength $L = 2\pi/k$ of the applied load. Consequently, there exists a sharp ratio between the horizontal and vertical length scales: $z/x = O(\epsilon)$, where

$$\epsilon \equiv kh \sim k\delta \ll 1$$

is a small ordering parameter. Second, the applied load $P$ has an amplitude that is one order of magnitude smaller than the hydrostatic pressure $\rho gh$, where $g$ is the acceleration due to gravity. As a result, the horizontal/vertical displacements of particles are also one order of magnitude smaller than the corresponding length scales. In other words, the induced wave amplitude is one order of magnitude smaller than the material layer thickness, which in turn is one order of magnitude smaller than the wavelength. We remark that such a
condition of shallowness and small displacements is often met in practice since the layer of fluidized mud on the bottom of an aquatic system is typically $O(10) \text{cm}$ thick [24], which is indeed much smaller than the wavelength of the surface wave. This is also the basis on which Ng [4] developed a two-layer Stokes boundary layer model for mud under waves.

Based on the considerations above, we may set out the following orders for the key variables of the present problem:

$$
(x, z) = O(k^{-1}), \quad (z, \beta) = O(h) = O(ek^{-1}), \quad t = O(\sigma^{-1}),
$$

$$
p = O(\rho h), \quad P_s = O(\epsilon \rho h),
$$

$$
(\tau_{xx}, \tau_{zz}) = O(\mu \sigma), \quad \tau_{xz} = O(\epsilon^{-1} \mu \sigma),
$$

where $p$ is the pressure and $\tau_{ij}$ are the stress components. Also, the horizontal and vertical displacements have the orders: $x - z = O(ek^{-1})$, $z - \beta = O(\epsilon h) = O(\epsilon^2 k^{-1})$.

Following Pierson [22], the governing equations of motion in Lagrangian form can be expressed as follows. For an incompressible material, the mass conservation is governed by

$$
\frac{\partial(x, z)}{\partial(x, \beta)} = 1.
$$

The horizontal and vertical momentum equations are, respectively, given by

$$
\frac{\partial^2 x}{\partial t^2} = - \frac{1}{\rho} \frac{\partial p(x, z)}{\partial(x, \beta)} + \frac{1}{\rho} \left[ \epsilon^2 \frac{\partial (\tau_{xx}, z)}{\partial(x, \beta)} + \frac{\partial (x, \tau_{xz})}{\partial(x, \beta)} \right],
$$

and

$$
\epsilon^2 \frac{\partial^2 z}{\partial t^2} = - \frac{1}{\rho} \frac{\partial p(x, \beta)}{\partial(x, \beta)} - g + \frac{\epsilon^2}{\rho} \left[ \frac{\partial (\tau_{xx}, z)}{\partial(x, \beta)} + \frac{\partial (x, \tau_{xz})}{\partial(x, \beta)} \right],
$$

where $\epsilon$ has been inserted for identification of the order of the associated term.

To solve the problem, we further need a constitutive law to relate the stress and the strain. In this study, we adopt the widely used Voigt model (e.g., [10,12,13,18,16]) to describe the viscoelastic behavior of the material. When described by the Voigt model, the stress is a linear combination of viscous and elastic components, which are, respectively, proportional to the strain rate and the strain. In this model, the viscous and the elastic parts will share the same deformation. Unlike a Newtonian fluid, a viscoelastic material cannot undergo continuous and infinite deformation when subject to a constant shear stress. One can envisage that the material responds more like a fluid at the outset of motion, but more like a solid ultimately when the elastic stress takes over the viscous stress and the motion stops. The present problem involves oscillatory back-and-forth motion on the first order, and net mean motion on the second order. Therefore, one may expect that the mean motion will die out asymptotically as the stress is completely handled by the elasticity of the material.

Here, we assume that the viscoelastic material is isotropic where the elastic part of the stress tensor is simply a linear function of the Lagrangian finite-strain tensor (e.g., [25,26]), which is given by

$$
E_{ij}^e = \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_m}{\partial X_i} \frac{\partial u_m}{\partial X_j},
$$

where $u_i$ is the displacement vector of a particle that is originally positioned at $X_i$. It follows that we may express the Lagrangian stress components for a Voigt body as follows:

$$
\tau_{xx} = 2\mu \frac{\partial (x, z)}{\partial(x, \beta)} + G \left[ 2 \frac{\partial (x - z)}{\partial x} \frac{\partial (x - z)}{\partial x} + \frac{\partial (x - z)}{\partial x} \frac{\partial (x - \beta)}{\partial x} + \epsilon^2 \frac{\partial (x - \beta)}{\partial x} \frac{\partial (x - \beta)}{\partial x} \right],
$$

$$
\tau_{zz} = 2\mu \frac{\partial (x, z)}{\partial(x, \beta)} + G \left[ 2 \frac{\partial (x - \beta)}{\partial \beta} + \frac{1}{\epsilon^2} \frac{\partial (x - z)}{\partial \beta} \frac{\partial (x - z)}{\partial \beta} + \frac{\partial (x - \beta)}{\partial \beta} \frac{\partial (x - \beta)}{\partial \beta} \right],
$$

and
\[ \tau_{xz} = \tau_{x\beta} \]
\[ = \mu \left[ \frac{\partial (x, \beta)}{\partial (x, \beta)} + \epsilon^2 \frac{\partial (z, \beta)}{\partial (x, \beta)} \right] \]
\[ + G \left[ \frac{\partial (x - \beta)}{\partial \beta} + \epsilon^2 \frac{\partial (z - \beta)}{\partial \beta} \right. 
\[ \left. + \frac{\partial (x - \beta)}{\partial x} \frac{\partial (x - \beta)}{\partial x} + \epsilon^2 \frac{\partial (z - \beta)}{\partial x} \frac{\partial (x - \beta)}{\partial x} \right] \] \tag{12}

where the overdot denotes time derivative, and \( G \) is the modulus of rigidity.

The above set of governing equations is to be solved with the following boundary conditions. On the free surface \((\beta = 0)\), the shear stress is zero, and the normal stress is to balance an applied load that is a periodic function of space and time given by
\[ p(\beta = 0^+) = -P_s \cos(kx - \sigma t), \tag{13} \]
where \( k = 2\pi / L \) is the wavenumber, \( \sigma = 2\pi / T \) is the angular frequency, and \( L \) and \( T \) are the wavelength and period, respectively, of the forcing. Accordingly, the dynamic free-surface boundary conditions can be written as [21]
\[ e^2 (\tau_{xx} - \tau_{x\beta}) \frac{\partial x}{\partial x} \frac{\partial z}{\partial x} + \tau_{x\beta} \left[ \frac{\partial (x^2)}{\partial x} - e^2 \left( \frac{\partial z}{\partial x} \right)^2 \right] = 0 \quad \text{at} \quad \beta = 0, \tag{14} \]

and
\[ -P \left[ \left( \frac{\partial x}{\partial x} \right)^2 + e^2 \left( \frac{\partial z}{\partial x} \right)^2 \right] + e^4 \tau_{xx} \left( \frac{\partial z}{\partial x} \right)^2 + e^2 \tau_{x\beta} \left( \frac{\partial x}{\partial x} \right)^2 - 2e^2 \tau_{xz} \frac{\partial x}{\partial x} \frac{\partial z}{\partial x} \]
\[ = -eP \left[ \left( \frac{\partial x}{\partial x} \right)^2 + e^2 \left( \frac{\partial z}{\partial x} \right)^2 \right] \cos(kx - \sigma t) \quad \text{at} \quad \beta = 0. \tag{15} \]

The free-surface elevation can be found from
\[ \eta(x, t) = z(x, 0, t). \tag{16} \]

It is assumed that the bed beneath the viscoelastic layer is rigid and impermeable. Therefore, on the bottom of the layer \((\beta = -h)\), the particle displacements and velocities are zero
\[ x = z = \dot{x} = \dot{z} = 0 \quad \text{at} \quad \beta = -h. \tag{17} \]

Following Pierson [22], we expand the variables into powers of \( \epsilon \):
\[ (x, z) = (x, \beta) + \epsilon (x_1, z_1) + \epsilon^2 (x_2, z_2) + \cdots, \tag{18} \]
\[ p = -\rho g \beta + \epsilon p_1 + \epsilon^2 p_2 + \cdots, \tag{19} \]
\[ \eta = \epsilon \eta_1 + \epsilon^2 \eta_2 + \cdots, \tag{20} \]
\[ \tau_{xz} = \tau_{x\beta} + \epsilon \tau_{x\beta 1} + \cdots, \tag{21} \]
\[ (\tau_{xz}, \tau_{x\beta}) = \epsilon (\tau_{x\beta 1}, \tau_{x\beta 1}) + \epsilon^2 (\tau_{x\beta 2}, \tau_{x\beta 2}) + \cdots, \tag{22} \]

where
\[ \tau_{x\beta 0} = G \left( \frac{\partial x_1}{\partial \beta} \right)^2, \tag{23} \]
\[ \tau_{x\beta 1} = 2\mu \left( \frac{\partial z_1}{\partial \beta} + 2G \left( \frac{\partial z_1}{\partial \beta} + \frac{\partial x_1}{\partial \beta} \frac{\partial x_2}{\partial \beta} \right) \right), \tag{24} \]
\[ \tau_{x\beta 2} = \mu \left( \frac{\partial x_1}{\partial \beta} + G \frac{\partial x_1}{\partial \beta} \right), \tag{25} \]
\[ \tau_{xz2} = \mu \left( \frac{\partial \ddot{x}_2}{\partial \beta} + \frac{\partial \dot{x}_1}{\partial \beta} - \frac{\partial x_1}{\partial \beta} \frac{\partial \ddot{x}_1}{\partial x} \right) + G \left( \frac{\partial \ddot{x}_2}{\partial \beta} + \frac{\partial \dot{x}_1}{\partial \beta} \frac{\partial \ddot{x}_1}{\partial x} \right), \]  
\[ \tau_{xz1} = 2\mu \frac{\partial \ddot{x}_1}{\partial x} + 2G \frac{\partial \dot{x}_1}{\partial x}, \]  
\[ \tau_{xz2} = 2\mu \left( \frac{\partial \ddot{x}_2}{\partial x} + \frac{\partial \dot{x}_1}{\partial x} \frac{\partial \ddot{x}_1}{\partial x} - \frac{\partial x_1}{\partial x} \frac{\partial \ddot{x}_1}{\partial x} \right) + G \left[ 2 \frac{\partial \ddot{x}_2}{\partial x} + \left( \frac{\partial \dot{x}_1}{\partial x} \right)^2 \right]. \]

Substitution of these expansions into the governing Eqs. (6)–(8) and the boundary conditions (14)–(17), and collecting terms of equal power of \( \epsilon \), we may obtain the perturbation equations at successive orders. Solutions to the \( O(\epsilon) \) and \( O(\epsilon^2) \) problems are presented in the next two sections.

### 3. First-order problem

At \( O(\epsilon) \), (6)–(8) give

\[ \frac{\partial x_1}{\partial x} + \frac{\partial z_1}{\partial \beta} = 0, \]  
\[ \frac{\partial^2 x_1}{\partial t^2} = -\frac{1}{\rho} \left[ \frac{\partial p_1}{\partial x} + \rho g \frac{\partial z_1}{\partial x} \right] + \frac{1}{\rho} \frac{\partial \tau_{xz1}}{\partial x}, \]

in which \( \tau_{xz1} \) is given by (25), and

\[ 0 = -\frac{1}{\rho} \left[ \frac{\partial p_1}{\partial \beta} - \rho g \frac{\partial x_1}{\partial x} \right]. \]

Eqs. (29)–(31) are the leading-order equations for the conservation of mass, and the conservation of momentum in the horizontal and vertical directions, respectively. Because of shallowness, the shear stress \( \tau_{xz1} \) dominates in the horizontal acceleration of particles, while in the vertical direction the pressure distribution is essentially hydrostatic.

The first-order boundary conditions include

\[ x_1 = z_1 = 0 \quad \text{at } \beta = -h, \]  
\[ \tau_{xz1} = 0 \quad \text{at } \beta = 0, \]  
\[ p_1 = P_s \cos(kx - \alpha t) \quad \text{at } \beta = 0, \]

and

\[ \eta_1(x, t) = z_1 \quad \text{at } \beta = 0. \]

To seek the solution, we let \( x_1, z_1, \) and \( p_1 \) have the same simple harmonic form as the forcing

\[ (x_1, z_1, p_1) = \text{Re} \left[ (\tilde{x}, \tilde{z}, \tilde{p}) e^{i(kx - \alpha t)} \right], \]

where Re stands for the real part, \( i \) is the complex unit, and \( \tilde{x}, \tilde{z}, \) and \( \tilde{p} \) are complex functions of \( \beta \) only. Substitution of (36) into (29)–(34) gives us

\[ \tilde{x} = ik^{-1} \tilde{z}, \]  
\[ -\sigma^2 \tilde{x} = -\frac{ik}{\rho} [\tilde{p} + \rho g \tilde{z}] + \left( G - i \mu \sigma \right) \tilde{x}'' \]  
\[ 0 = -\frac{1}{\rho} [\tilde{p}' + \rho g \tilde{z}'], \]  
\[ \tilde{x} = \tilde{z} = 0 \quad \text{at } \beta = -h, \]  
\[ \tilde{x}' = 0 \quad \text{at } \beta = 0, \]
\[ \tilde{p} = P_s \quad \text{at} \quad \beta = 0, \tag{42} \]
in which the prime denotes differentiation with respect to \( \beta \). Eqs. (37)–(39) can be combined to yield
\[ \tilde{z}'''' + \lambda^2 \tilde{z}'' = 0, \tag{43} \]
where \( \lambda \) is a complex parameter given by
\[ \lambda^2 = \frac{\rho \sigma^2}{G - i \mu \sigma} = (\lambda_e^{-2} + \lambda_v^{-2})^{-1}, \tag{44} \]
in which \( \lambda_e \) and \( \lambda_v \) are, respectively, related to the elasticity and viscosity of the material, as given by
\[ \lambda_e^2 = \frac{\rho \sigma^2}{G}, \tag{45} \]
and
\[ \lambda_v^2 = i \rho \sigma / \mu = (1 + i)^2 / \delta^2. \tag{46} \]
Hence, \( \lambda_e \) is the wavenumber of the elastic transverse wave across the layer, and \( \lambda_v \) is inversely proportional to the Stokes boundary layer thickness \( \delta = \sqrt{2 \nu / \sigma} \) arising from an oscillatory viscous boundary layer. We are here primarily interested in a soft material that has a sufficiently low stiffness/viscosity so that it can readily deform/flow in response to a driving force. Specifically, we assume that \( G \) and \( \mu \) are limited in value such that \( \lambda_e \) and \( \lambda_v \) are both of \( O(h^{-1}) \) or greater. Of course, the limiting values \( G = 0 \) (\( \lambda = \lambda_v \)) or \( \mu = 0 \) (\( \lambda = \lambda_e \)) correspond to the cases when the material is purely viscous or elastic, respectively.

The general solution to (43) can be written as
\[ \tilde{z}(\beta) = A(h + \beta) + B + C \sin \lambda(h + \beta) + D \cos \lambda(h + \beta), \tag{47} \]
where the first two terms arise from the inviscid (i.e., when both \( G \) and \( \mu \) vanish) component of the material motion, while the last two terms correspond to the component of motion that is controlled by the viscoelasticity of the material. The constants \( A, B, C, \) and \( D \) in the solution above can be determined from the boundary conditions (40)–(42). With some algebra, we may finally obtain the first-order solution as follows:
\[ \tilde{z}(\beta) = C[-\lambda(h + \beta) + \tan \lambda h + \sin \lambda(h + \beta) - \tan \lambda h \cos \lambda(h + \beta)], \tag{48} \]
\[ \tilde{x}(\beta) = i C k^{-1} \lambda [1 + \cos \lambda(h + \beta) + \tan \lambda h \sin \lambda(h + \beta)], \tag{49} \]
\[ \tilde{p}(\beta) = -C \rho g \left[ (\sigma^2 / g k^2 h) \lambda h - \lambda(h + \beta) + \tan \lambda h + \sin \lambda(h + \beta) - \tan \lambda h \cos \lambda(h + \beta) \right], \tag{50} \]
where
\[ C = \frac{-P_s / \rho g}{(\sigma^2 / g k^2 h)(\lambda h - \lambda h + \tan \lambda h)}. \tag{51} \]
In the limit of a pure viscous material (i.e., \( G = 0 \) or \( \lambda = \lambda_v \)), the solutions (48)–(51) are identical to the solutions for a Newtonian fluid that have been deduced by Ng [21]. In the limit of a pure elastic material (i.e., \( \mu = 0 \), \( \lambda = \lambda_e \)) becomes a real number, and propagating across the layer is an elastic wave of wavenumber \( \lambda \), which is a classical phenomenon [27] that has been extensively examined in the context of waves over a poro-elastic bed (e.g., see [28] and the references therein). Despite a one-layer system being considered here, the first-order solutions derived above can describe qualitatively the same wave phenomena (e.g., occurrence of resonance) as in a two-layer system, as have been studied by Hsiao and Shemdin [9], MacPherson [10], and Piedra-Cueva [13].

4. Second-order problem

At \( O(\epsilon^2) \), the governing equations for the conservation of mass, and the conservation of momentum in the horizontal and vertical directions are, respectively, given by
\[ \frac{\partial x_2}{\partial x} + \frac{\partial z_2}{\partial \beta} = -\frac{\partial x_1}{\partial z} \frac{\partial z_1}{\partial \beta} + \frac{\partial x_1}{\partial \beta} \frac{\partial z_1}{\partial x}, \tag{52} \]
\[
\frac{\partial^2 x_2}{\partial t^2} = -\frac{1}{\rho} \left[ \frac{\partial p_z}{\partial x} + \rho g \frac{\partial z_2}{\partial x} + \frac{\partial p_1}{\partial \beta} \frac{\partial c_1}{\partial x} - \frac{\partial p_1}{\partial \beta} \frac{\partial c_1}{\partial x} \right] + \frac{1}{\rho} \left[ \frac{\partial \tau_{zx_2}}{\partial \beta} + \frac{\partial \tau_{zx_1}}{\partial \beta} \frac{\partial c_1}{\partial \beta} - \frac{\partial \tau_{c_1}}{\partial \beta} \frac{\partial c_1}{\partial \beta} \right],
\] (53)

and
\[
0 = -\frac{1}{\rho} \left[ \frac{\partial p_z}{\partial x} + \rho g \frac{\partial z_2}{\partial x} + \frac{\partial p_1}{\partial \beta} \frac{\partial c_1}{\partial x} - \frac{\partial p_1}{\partial \beta} \frac{\partial c_1}{\partial x} \right] + \frac{1}{\rho} \frac{\partial \tau_{zx_2}}{\partial \beta},
\] (54)

in which the stress components \( \tau_{zx_0}, \tau_{zx_1}, \) and \( \tau_{zx_2} \) have been given in (23), (25), and (26), respectively.

Boundary conditions of \( O(e^2) \) read as follows:
\[
x_2 = z_2 = \dot{x}_2 = \dot{z}_2 = 0 \quad \text{at} \ \beta = -h,
\] (55)
\[
\tau_{zx_2} = 0 \quad \text{at} \ \beta = 0,
\] (56)
\[
p_2 = 0 \quad \text{at} \ \beta = 0,
\] (57)

and
\[
\eta_2(z,t) = z_2 \quad \text{at} \ \beta = 0.
\] (58)

Our interest here is to find the time-mean motion, which can be accomplished by taking time-average of the equations above. We first note that the second-order mean motion arises from the forcing terms composed of products of the first-order functions. Since a non-decaying progressive wave is considered, the mean motion will be uniform in the horizontal direction. Therefore the \( x \)-derivative of any time-averaged forcing term will be zero. In particular, we have
\[
\frac{\partial}{\partial x} (\bar{p}_2 + \rho \bar{g} \bar{z}_2) = 0, \quad \frac{\partial \bar{\tau}_{zx_0}}{\partial x} = 0,
\] (59)

where the overbar denotes time-averaging over one wave period. On the substitution of (25) and (26) for \( \tau_{zx_1} \) and \( \tau_{zx_2} \), the time-average of (53) will then yield the following governing equation for the second-order mean displacement \( x_2 \):
\[
\frac{\partial^2}{\partial \beta^2} (\mu \ddot{x}_2 + G \dddot{x}_2) = E,
\] (60)

where \( E = E(\beta) \) given by
\[
E = \frac{\partial p_1}{\partial \beta} \frac{\partial c_1}{\partial \beta} - \frac{\partial p_1}{\partial \beta} \frac{\partial c_1}{\partial \beta} \frac{\partial c_1}{\partial \beta} - \frac{\partial p_1}{\partial \beta} \frac{\partial c_1}{\partial \beta} \frac{\partial c_1}{\partial \beta} + \frac{\partial^2 (\mu \ddot{x}_1 + G \dddot{x}_1)}{\partial \beta^2} \frac{\partial c_1}{\partial \beta}.
\] (61)

We remark that on deriving (60), we have ignored the time-mean of the acceleration term of the momentum equation. This is justified if the time-mean motion has a constant or very slowly varying velocity (Lagrangian drift or mass transport velocity). Such a creeping motion can result from low stiffness of the material. As is shown below, this condition is met when \( G/\mu \ll \sigma \).

On integrating (60) twice with respect to \( \beta \), and making use of the time-averaged boundary conditions (55) and (56), we obtain
\[
\mu \ddot{x}_2 + G \dddot{x}_2 = H,
\] (62)

where \( H = H(\beta) \) is given by
\[
H = \int_0^\beta \int_0^\beta E \, d\beta \, d\beta.
\] (63)

Eq. (62) can readily be solved to give us the desired second-order mean motion of the material. Three cases can be identified.
1. If $G = 0$ and the material is a pure viscous fluid, (62) gives a steady mass transport velocity

$$v_L = \dot{x}_2 = H/\mu,$$

which corresponds to the Newtonian case examined by Ng [21].

2. If $\mu = 0$ and the material is a pure elastic solid, (62) gives a steady displacement

$$\bar{x}_2 = H/G.$$

One can however show that in this case ($\delta = 0$) $\bar{z}$ and $\bar{p}$ are real and $\bar{x}$ is imaginary. It follows that $E$ is identically zero, and hence so is $H$. In other words, the net displacement will be exactly zero for an elastic medium without viscosity.

3. If the material is in general viscoelastic (i.e., $\mu \neq 0$ and $G \neq 0$), (62) gives a time decaying mean motion

$$\bar{x}_2(t) = \frac{H}{G}[1 - e^{-Gt/\mu}].$$

Hence, the net displacement will tend to the steady limit $H/G$ over a timescale $\mu/G$. To justify our earlier assumption of a slow or creeping mean motion, this timescale must be much longer than the wave period: $G/\mu \ll \sigma$ or the material must not be too stiff. If this condition is not met, the finite deformation can be attained rapidly and the transient motion is of no significance at all.

Let us remark again that a periodic driving force, when applied to a sufficiently soft viscoelastic layer, is to result in a creeping mean motion of the viscoelastic material as described by (66); this result appears not to have been reported previously in the literature. We further note that during small times such that $Gt/\mu = 1$, the creeping mean motion is at a velocity approximately given by the mass transport velocity (64). This explains why Lian et al. [19], who considered a rather soft viscoelastic estuarine mud with $G/\sigma \mu = 0.01$, could compare their calculated mass transport velocity with some experimental data. Of course, such a seemingly good comparison works only for the initial stage of the mud motion; upon a long enough time, the creeping mean motion will eventually die out. It is important to realize that the mass transport undergone by a viscoelastic material can only be a time-dependent process.

### 5. Normalization

To facilitate numerical discussions, let us first introduce the following normalized variables, which are distinguished by a caret:

$$\begin{align*}
(x, x_1, x_2) &= k^{-1}(\dot{x}, \dot{x}_1, \dot{x}_2), \\
(\beta, z_1, \delta) &= h\left(\dot{\beta}, \dot{z}_1, \dot{\delta}\right), \\
(p_1, P_s) &= \epsilon \rho g h(p_1, P_s), \\
(\lambda, \lambda_e, \lambda_v) &= h^{-1}(\dot{\lambda}, \dot{\lambda}_e, \dot{\lambda}_v).
\end{align*}$$

where

$$\lambda = \left(\dot{\lambda}_e^{-2} + \dot{\lambda}_v^{-2}\right)^{-1} = \left(\dot{\lambda}_e^{-2} - \frac{i\delta^2}{2}\right)^{-1}.$$  

The normalized first-order solution can be written as follows:

$$\begin{align*}
(\dot{x}_1, \dot{z}_1, \dot{p}_1) &= \text{Re}\left[(\dot{\lambda}, \dot{\lambda}_e, \dot{\lambda}_v)e^{i(\beta - \delta)}\right],
\end{align*}$$

where

$$\begin{align*}
\dot{z}(\beta) &= \tilde{C}\left[-\dot{\lambda}(1 + \dot{\beta}) + \tan \dot{\lambda} \sin \dot{\lambda}(1 + \dot{\beta}) - \tan \dot{\lambda} \cos \dot{\lambda}(1 + \dot{\beta})\right], \\
\dot{\lambda}(\beta) &= i\tilde{C}\dot{\lambda}\left[-1 + \cos \dot{\lambda}(1 + \dot{\beta}) + \tan \dot{\lambda} \sin \dot{\lambda}(1 + \dot{\beta})\right], \\
\dot{p}(\beta) &= -\tilde{C}\dot{\lambda}(Fr - 1 - \dot{\beta}) + \tan \dot{\lambda} + \sin \dot{\lambda}(1 + \dot{\beta}) - \tan \dot{\lambda} \cos \dot{\lambda}(1 + \dot{\beta}).
\end{align*}$$
\[ Fr = \frac{\sigma^2}{gk^2h}, \]  
\[ \hat{C} = \frac{-\dot{P}_s}{\lambda(Fr - 1) + \tan \hat{\lambda}}. \]

The first-order free-surface elevation is

\[ \hat{\eta}_1 = \text{Re} \left[ \hat{C} \left( \tan \hat{\lambda} - \hat{\lambda} \right) e^{(\hat{\lambda}-\hat{\eta})} \right]. \]

In summary, we need four input parameters in order to calculate the dimensionless first-order solution: \( Fr \) for the propagating speed of the applied pressure wave, \( P_s \) for the amplitude of the loading, \( \delta \) for the viscous Stokes boundary layer thickness, and \( \lambda_e \) for the wavenumber of the elastic transverse wave. Note that \( \lambda_e \) is inversely proportional to \( G \), while \( \delta \) is proportional to the square root of \( v \). Hence, the material will be purely viscous when \( \lambda_e = \infty \), and will be purely elastic when \( \delta = 0 \). The material will be more rigid for smaller \( \lambda_e \), and will be more viscous for larger \( \delta \).

The motion will be finite in amplitude as long as the denominator of \( \hat{C} \) does not vanish, which is true as far as \( \hat{\lambda} \) is complex or the viscosity is non-zero. When the material is purely elastic for which \( \hat{\lambda} \) becomes real, it is however possible to find admissible values of \( \hat{\lambda} \) and \( Fr \) satisfying

\[ \hat{\lambda}(Fr - 1) + \tan \hat{\lambda} = 0, \]

which corresponds to blow-up of the solution, or resonance of the system. For example, when \( Fr = 1 \), the equation above is satisfied by \( \hat{\lambda} = \hat{\lambda}_e = \pi, 2\pi, 3\pi, \ldots \) That is, an elastic layer can be excited to resonance when the forcing frequency is to cause the transverse wavelength to be a certain multiple of the layer thickness. The viscosity, if present, will offer damping and hence no perfect blow-up will happen, but the motion may still be amplified at some resonant frequencies of the forcing depending on the viscosity; this is called practical resonance.

It is also of interest to note that, for a pure elastic medium, the first-order free-surface elevation can be identically zero, \( \hat{\eta}_1 \equiv 0 \), when \( \tan \hat{\lambda} = \hat{\lambda}_e \). This happens despite the interior displacement being non-zero. We shall further look into this phenomenon in the next section.

For the second-order mean motion, we further introduce the following normalization:

\[ (E, H) = \hat{c}^2 k^{-1} \rho \sigma^2 (\hat{E}, \hat{H}). \]

Then (62) can be written as

\[ \left( \dot{\hat{\delta}}^2/2 \right) \hat{x}_2 + \left( \hat{\lambda}_e^{-2} \right) \hat{x}_2 = \hat{H}, \]

where

\[ \hat{H} = \int_{-\beta}^{\beta} \int_{0}^{\beta} \hat{E} \, d\hat{\beta} \, d\hat{\beta}, \]

and

\[ \hat{E} = Fr^{-1} \left( \frac{\partial p_1}{\partial \hat{x}} \, \frac{\partial \hat{\xi}_1}{\partial \hat{\beta}} - \frac{\partial p_1}{\partial \hat{\beta}} \, \frac{\partial \hat{\xi}_1}{\partial \hat{x}} \right) - \frac{\dot{\hat{\delta}}^2}{2} \, \frac{\partial}{\partial \hat{\beta}} \left( \frac{\partial \hat{x}_1}{\partial \hat{x}} \, \frac{\partial \hat{x}_1}{\partial \hat{\beta}} - \frac{\partial \hat{x}_1}{\partial \hat{\beta}} \, \frac{\partial \hat{x}_1}{\partial \hat{x}} \right) \]

\[ - \hat{\lambda}_e^{-2} \, \frac{\partial}{\partial \hat{\beta}} \left( \frac{\partial \hat{x}_1}{\partial \hat{x}} \, \frac{\partial \hat{x}_1}{\partial \hat{\beta}} \right) \]

\[ + \frac{\partial^2}{\partial \hat{\beta}^2} \left( \frac{\partial^2 \hat{x}_1}{\partial \hat{\beta} \partial \hat{\beta}} \right) \]

\[ + \hat{\lambda}_e^{-2} \hat{\lambda}_e^{-2} \frac{\partial \hat{x}_1}{\partial \hat{\beta}} \frac{\partial \hat{x}_1}{\partial \hat{\beta}} \]

The mean displacement given by (66) then becomes
which tends to the following limit of steady deformation at large times:

\[ \hat{x}_2^\infty(\beta) = \lambda_\varepsilon^2 \hat{H} \left[ 1 - \exp \left( -\frac{2\beta}{\lambda_\varepsilon^2 \delta^2} \right) \right], \]  

(81)

or when the elasticity vanishes, the mean motion is a steady drift with a constant mass transport velocity

\[ \hat{v}_L = 2\hat{H}/\delta^2 \quad \text{as} \quad \lambda_\varepsilon \to \infty. \]  

(83)

6. Results and discussions

Some numerical results are presented in this section, allowing us to further look into the effects of viscoelasticity on the first-order oscillatory motion and the second-order mean displacement of the medium. In the cases discussed below, the input values of \( Fr = \tilde{P}_e = 1.0 \) have been used throughout. We remark that, irrespective of the material rheology, the first- and second-order displacements are always proportional to the first and second powers of \( \tilde{P}_e \), respectively, and therefore it suffices to consider a unity value of this parameter for the pressure load. This is unlike the case of power-law fluid considered by Ng [21], in which the effect of the surface pressure on the fluid motion is a function of the power-law index, and is therefore sensitive to the rheology.

To check our calculations, we have compared some results for the pure viscous case with the corresponding Newtonian results obtained by Ng [21], and found that they agree with each other with very good accuracy.

We first show in Fig. 1 the first-order free-surface elevation, \( \hat{\eta}_1 \), as a function of the phase \( \xi = \hat{x} - t \), and the parameters \( \lambda_\varepsilon \) and \( \tilde{\delta} \). This figure should be examined in conjunction with Fig. 2, which shows the amplitude of the free-surface wave, \( |\hat{\eta}_1| \), as a function of \( \lambda_\varepsilon \) and \( \tilde{\delta} \). Except for the pure elastic case, the phase difference between the forcing and the free-surface wave varies with both \( \lambda_\varepsilon \) and \( \tilde{\delta} \). When it is purely elastic, the peak of \( \hat{\eta}_1 \) occurs at \( \xi = \pi \), and when the viscosity dominates, the peak occurs in the range \( \pi/4 < \xi < \pi/2 \). More important, the amplitude of the free-surface wave varies in a more remarkable fashion with the parameters. For the pure elastic case (\( \tilde{\delta} = 0 \)), as noted earlier, resonance occurs when \( \lambda_\varepsilon = \pi, 2\pi, \ldots \), and the free-surface wave amplitude is identically zero when \( \lambda_\varepsilon = 4.49341, 7.72525, \ldots \), which are the roots of \( \tan(\lambda_\varepsilon) = \lambda_\varepsilon \). The alternate occurrence of these two phenomena explains the dramatic change in the amplitude of \( \hat{\eta}_1 \) for \( \tilde{\delta} = 0 \), as shown in Fig. 2. The amplitude may change abruptly from zero to infinity, or vice versa, over a small range of \( \lambda_\varepsilon \). A medium with sufficiently small viscosity (\( \tilde{\delta} = 0.1 \)) is still much controlled by elasticity, and will be subjected to practical resonance at the same values of \( \lambda_\varepsilon = \pi, 2\pi, \ldots \), where the amplification factor is the largest at the primary resonance frequency \( \lambda_\varepsilon = \pi \). As the material becomes more viscous so that \( \tilde{\delta} \geq 0.5 \), elasticity will primarily lose its influence on the motion of the material, except at the lower limit of \( \lambda_\varepsilon \) corresponding to a very stiff material.

The displacement of particles within the layer as a function of the phase \( 0 < \xi < 2\pi \) is shown in the two-dimensional vector plots of Figs. 3–5, for \( \tilde{\delta} = 0, 0.1, 0.3 \), respectively. The vector lengths have been plotted according to the ratio of 0.01 grid units per magnitude. In the pure elastic case of \( \tilde{\delta} = 0 \) (Fig. 3), the particle displacements are extraordinarily large at the near-resonance values of \( \lambda_\varepsilon = 3, 6 \), but become diminished at \( \lambda_\varepsilon = 4.49341, 7.72525 \) when the free-surface wave amplitude is zero. The latter cases show that a zero free-surface elevation does not imply no motion across the whole layer; it happens because the particles move only horizontally on the free surface. There are also cellular circulatory patterns featured in these vector plots. These patterns arise when there are planes within the layer on which the particles move only horizontally, vertically, or along any other fixed inclination. The number of cells stacking across the layer increases with \( \lambda_\varepsilon \), or with a decreasing wavelength of the transverse wave. It is remarkable that, when a cellular pattern emerges, it is no longer necessary for the displacement amplitude to be the maximum on the free surface; the displacement of an interior particle can be larger in amplitude than that on the top (Fig. 3d). It appears, however, that when a cellular pattern shows up clearly in the displacement field, the motion within the layer is on the whole diminished.
The second case with non-zero but weak viscosity of \( \hat{\delta} = 0.1 \) (Fig. 4) has displacement fields that to some extent look qualitatively the same as the those in the pure elastic case (Fig. 3). A cellular pattern emerges at \( \hat{\lambda}_e = 5 \) (Fig. 4b), when the free-surface elevation amplitude is close to its first local minimum. No cellular pattern however shows up at \( \hat{\lambda}_e = 8 \) (Fig. 4c), when the free-surface elevation amplitude is close to the second local minimum; this is a sign suggesting that by this stage the viscous damping becomes comparable with the elastic effects. It is interesting to compare the two cases of practical resonance. When \( \hat{\lambda}_e = 3 \) (Fig. 4a), particles especially those near the top of the layer are excited to move with a very large amplitude, and it is always the free-surface particles that have the largest horizontal displacement. In contrast, when \( \hat{\lambda}_e = 10 \) (Fig. 4d), the horizontal displacement does not increase monotonically with height above the bottom, and the particles with the maximum horizontal displacement are actually located near the bottom (\( \beta \approx -0.7 \)) of the layer.

Fig. 1. First-order free-surface elevation \( \hat{\eta}_1(\xi) \) as a function of \( \hat{\lambda}_e \): (a) \( \hat{\delta} = 0 \) (pure elastic); (b) \( \hat{\delta} = 0.1 \); (c) \( \hat{\delta} = 0.3 \); (d) \( \hat{\delta} = 0.5 \).
Although the resonance amplification is the strongest for \( \lambda_e = 3 \) (Fig. 4a), the particles near the bottom in this case do not move as much as those in the other cases. The third case with higher viscosity of \( \delta = 0.3 \) (Fig. 5) shows more regular displacement fields: the displacement amplitude increases monotonically with height above the bottom so that it is always the maximum on the top of the layer. Resonance amplification, although much damped by viscosity, still happens at \( \lambda_e = \pi \). Therefore, the displacement amplitude is larger for \( \lambda_e = 3 \) (Fig. 5a) than when the elasticity vanishes (i.e., pure viscous; Fig. 5b).

Let us now turn our attention to the second-order mean displacement \( \hat{x}_2(\hat{g}, \hat{i}) \), which is given by (81). Fig. 6 shows the time evolution of the mean displacement profiles for \( \lambda_e = 10 \), and \( \delta = 1, 0.5 \), which correspond to rather viscous materials. Two points are noteworthy here. First, it takes a longer time to approach the ultimate steady profile for the material with higher viscosity (Fig. 6a). This is consistent with (82) discussed above. Second, the steady deformation is larger in magnitude for the material with smaller viscosity (Fig. 6b). From these two points, we may infer that, for a sufficiently viscous material, the viscosity is to retard and to diminish the magnitude of both the first-order oscillatory and the second-order mean motions. This statement is however not true for a material of low viscosity; see below.

Fig. 7 shows some profiles of the steady mean displacement \( \hat{x}_2^\omega \), which is given by (82), for several values of \( \delta \) and \( \lambda_e \). The following observations can be made based on this and the earlier figures.

1. We first recall that the mean displacement is identically zero for a pure elastic material. In the absence of damping, particles of an elastic body will only undergo perfectly closed-path oscillatory motions. Elasticity enables the body to recover from any deformation when the applied force is removed, while viscosity will spoil such restoring capacity. For a pure elastic material, no matter how large its first-order oscillatory motion is, or even when it is in resonance, its second-order mean displacement still remains identically zero. The presence of even small viscosity will however dramatically change the nature of the consequence, as is evident in Fig. 7a. For \( \delta \) as low as 0.1, the mean displacement can be very substantial in extent, provided that the material is not too rigid (i.e., \( \lambda_e > 1 \)).

2. As there is a factor of \( \lambda_e^2 \) in the expression for \( \hat{x}_2^\omega \) given by (82), one may expect that the steady mean displacement is a strong function of \( \lambda_e \). This is confirmed by Fig. 7. In general, a larger value of \( \lambda_e \) (or smaller \( G \), corresponding to a softer material) will lead to larger deformation. It is interesting to note that the large-amplitude oscillatory motion arising from resonance on the first order does not necessarily lead to a larger magnitude of mean deformation on the second order.
It is remarkable that, in the case of $d = 0.1$, the mean displacements for $k_e = 5, 8,$ and 10 are negative over the top part of the layer. These materials are subjected to a strong backward deformation gradient in the middle of the layer, causing the displacement to turn sharply from a positive maximum to a negative maximum value. Such a sharp change in mean deformation can be correlated with the cellular pattern and non-monotonic profiles exhibited in the first-order displacement field, as have been seen in Fig. 4. We recall that the horizontal component of the first-order displacement does not increase monotonically with height in the two cases, $k_e = 8$, and 10 as shown in Figs. 4c and d; the peak value is located near $\beta = -0.7$ while a local minimum is located near $\beta = -0.3$. Apparently, these positions for the local extreme values correspond to where there is a sharp gradient in the mean deformation profile.

Fig. 3. Displacement vectors $(x, z)$ as a function of depth $\beta$ and phase $\zeta$ for $\delta = 0$ (pure elastic): (a) $\lambda_e = 3$; (b) $\lambda_e = 4.49341$; (c) $\lambda_e = 6$; (d) $\lambda_e = 7.72525$. 

3. It is remarkable that, in the case of $\delta = 0.1$, the mean displacements for $\lambda_e = 5, 8$, and 10 are negative over the top part of the layer. These materials are subjected to a strong backward deformation gradient in the middle of the layer, causing the displacement to turn sharply from a positive maximum to a negative maximum value. Such a sharp change in mean deformation can be correlated with the cellular pattern and non-monotonic profiles exhibited in the first-order displacement field, as have been seen in Fig. 4. We recall that the horizontal component of the first-order displacement does not increase monotonically with height in the two cases, $k_e = 8$, and 10 as shown in Figs. 4c and d; the peak value is located near $\beta = -0.7$ while a local minimum is located near $\beta = -0.3$. Apparently, these positions for the local extreme values correspond to where there is a sharp gradient in the mean deformation profile.
4. It has been inferred from Fig. 6 that, for a very viscous material, the viscosity is to diminish the motion of particles. Here the opposite effect is manifested by the two values of $\lambda_\varepsilon$, shown in Figs. 7b and c, when the material is sufficiently soft. These are cases in which the material is still relatively weak in viscosity, and increasing the viscosity will actually increase the magnitude of the mean deformation undergone by the material. This happens when the viscous effect becomes comparable with the restoring effect due to elasticity. Under this condition, the material will have a longer time to “flow” like a fluid before it feels the counter-balancing elastic force. This condition is effective until the viscosity becomes so large that the viscous force itself also retards the motion.

Finally, let us briefly compare the key findings of the present work with those of Ng [21], in order to highlight the differences arising from the viscoelastic and power-law rheologies. First, transverse elastic waves, and
Fig. 5. Displacement vectors \((\tilde{x}_1, \tilde{z}_1)\) as a function of depth \(\hat{\beta}\) and phase \(\xi\) for \(\hat{\delta} = 0.3\): (a) \(\hat{\lambda}_c = 3\); (b) \(\hat{\lambda}_c = \infty\) (pure viscous).

Fig. 6. Evolving profiles of the mean horizontal displacement \(\tilde{x}_2(\hat{\beta})\) as a function of time \(\hat{t}\) for \(\hat{\lambda}_c = 10\): (a) \(\hat{\delta} = 1\); (b) \(\hat{\delta} = 0.5\).
therefore resonance, can occur in a viscoelastic layer, but not in a power-law fluid layer. In this connection, it is distinct of a viscoelastic layer to possibly feature a non-monotonic particle displacement profile, or a displacement field with a cellular structure. Second, the applied load can cause non-trivial effects on a power-law fluid depending on the power-law index, while for a viscoelastic material the effect of the applied load is independent of the material rheology. Third, for a power-law fluid, the particle displacements are a non-linear function of space and time, and there can be motionless or plug-flow like regions where the shear level is low. No similar phenomena happen to a viscoelastic material. Fourth, the mean-motion of a viscoelastic material can be opposite in direction to the wave propagation, while the Lagrangian drift of a power-law fluid can only be in the same direction as the wave propagation.

Fig. 7. Ultimate steady profiles of the mean horizontal displacement $\tilde{x}_2 (\tilde{\beta})$ as a function of $\tilde{\chi}_c$: (a) $\tilde{d} = 0.1$; (b) $\tilde{d} = 0.3$; (c) $\tilde{d} = 0.5$. 

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7. Summary and concluding remarks

We have performed a perturbation analysis to the second order for the oscillatory and mean motions induced by a forced wave propagating in a thin viscoelastic layer. Compared with those in the literature, our analysis is distinct in the following aspects. First, in view of the fact that the displacement of a particle is by definition a Lagrangian quantity, the Lagrangian approach has been employed for the present problem. This approach is more appropriate than the commonly used Eulerian approach, since the latter works only for a linearized analysis involving small-amplitude displacements. Here second-order displacements need to be considered, and naturally the Lagrangian approach should be used. Second, we have avoided the mistake made by some previous authors, who have wrongly applied the complex viscoelastic parameter, which is actually valid for a first-order theory only, to the second-order analysis. We have pointed out that the second-order strain and strain rate are not as simply related to each other as their first-order counterparts. Third, we have shown that a viscoelastic material will not in general undergo a mean motion with a constant drift velocity, as has been assumed in previous studies. In fact the mean motion will decay exponentially over a timescale equal to the ratio of the viscosity to the shear modulus. Therefore, the mean deformation attained as the mean motion dies out will be as important as the mean motion itself to a viscoelastic material under wave action. Fourth, we have shown with numerical examples the possible effects due to the first-order oscillatory motion (e.g., resonance, cellular pattern) on the second-order steady mean deformation. The response of a viscoelastic material can be dramatically different from that of the either limiting case of a pure viscous or pure elastic material. For fixed viscosity, the steady mean deformation is typically larger for a softer material, regardless of the occurrence of resonance on the first order. The mean displacement tends to increase with height so that the maximum displacement is at or near the top of the layer, except when a cellular pattern shows up in the oscillatory displacement field. For fixed rigidity, there is an optimum value of viscosity at which the mean deformation is the largest. The mean deformation will diminish for a too large or too small value of viscosity. A pure elastic material will have identically zero mean deformation, no matter what happens to its oscillatory motion. A viscoelastic material that has both small but non-zero viscosity and elasticity can be subjected to a strong negative deformation gradient in the middle of the layer, resulting in a very large backward displacement across the upper part of the layer.

We have set out to investigate the mechanics of viscoelastic mud under the action of waves in a coastal environmental. Through a simplified scenario examined in this study, we have now developed insights into how to approach the full problem. A theory that takes into account the aspects mentioned above will be advanced as an extension of the present work for the mass transport in water waves over a thin layer viscoelastic mud. We shall report the results shortly.

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