Effective slip for Stokes flow over a surface patterned with two- or three-dimensional protrusions

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Abstract

Effective slip lengths are obtained, using semi-analytic methods, for Stokes flows over a surface that is patterned with a periodic array of two-dimensional (2D) cylindrical or 3D spherical protrusions. The protruding surface can be perfect- or non-slipping, corresponding to a bubble mattress or a rough boundary. For longitudinal and transverse flows over cylindrical bumps and 3D flow over a square array of spherical bumps, the effective slip length is obtained as a function of the protrusion angle, the area fraction of surface covered by protrusions and the partial slip length of the protruding surface. The results are compared with analytical dilute limits in order to ascertain the range of validity of these limits. Phenomenological equations are also derived to enable a quick evaluation of the slip length for some particular values of the protrusion angle at which the slip length is maximum in magnitude.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the context of microchannel flow, much work has been performed in recent years to study the effects of wall microstructures on the flow. Wall patterns will have effects on various aspects (e.g. velocity profile, direction and discharge rate) of the flow past them, thereby affecting associated phenomena such as hydrodynamic dispersion length and time scales. On the wall, a pattern can be two-dimensional (2D) or 3D in form, and the liquid boundary can be flat or curved and can be locally non-slipping, partial-slipping or even perfect-slipping. Wall structures are often used as a means of controlling the flow through a microchannel.

Heterogeneities on a wall can be viewed as roughness, as they tend to protrude into the fluid thereby retarding the flow near the wall. This results in enhanced wall friction. However,
such 'roughness' may, under some conditions, enhance the wall slippage instead (Cottin-Bizonne et al. 2003). A superhydrophobic surface is a hydrophobic surface (i.e. a dewetting surface with a contact angle larger than 90°), on which the roughness elements are so high and dense that the capillarity is strong enough to prevent a liquid from penetrating the minute cavities between the roughness elements, which are then occupied by a much less viscous gas phase. The microbubbles so trapped in the crevices will form a lubricating layer on the surface, leading to an effective slippery condition on the surface. Significant drag reduction in microchannels can be achieved using such superhydrophobic surfaces (Ou et al. 2004). The extent of wall slippage depends on a number of factors, such as the period of the pattern, shear-free area fraction, flow direction, channel height, curvature of the liquid–gas meniscus, intrinsic slippage, gas viscosity, Reynolds number and so on (Cottin-Bizonne et al. 2004, Ybert et al. 2007, Sbragaglia and Prosperetti 2007b, Maynes et al. 2007, Feuillebois et al. 2009, Cheng et al. 2009, Ng et al. 2010). See Rothstein (2010) for a recent review on slip over superhydrophobic surfaces.

The notion that microbubbles will always lead to wall slippage is not true either. Steinberger et al. (2007) presented experimental evidence that the gas trapped at a solid surface can act as an anti-lubricant and promote high friction if the bubbles protrude too much into the liquid. Despite their slippery surface, bubbles of a large protrusion will trap an immobile layer of liquid above the solid surface, which amounts to shifting the zero-velocity point into the fluid, thereby resulting in a negative slip length.

The possibility of negative effective slip over a bubble mattress was further investigated by Hyvälouma and Harting (2008), who conducted numerical simulations by a two-phase lattice Boltzmann model (Harting et al. 2010). For the flow over bubbles in circular holes that are arranged in a square, rectangular or rhombic array, they found a similar trend for the dropping of the effective slip length to a negative value at a specific protrusion angle of the bubbles. They also found a decrease in the effective slip length with increasing shear rate owing to deformation of the bubbles. More interesting are their findings for flow over 2D cylindrical bubbles in long slots. They found that the effective slip would strongly depend, with qualitatively different behaviors, on the flow direction. When the flow is parallel to the slots, the slip is positive, but for perpendicular flow the slip becomes negative. They reasoned that, in the former, the streamlines are straight and the flow does not ‘see’ any roughness but can feel the slippery bubble surface, while in the latter, the bubbles act as a barrier impeding the flow near them. Hyvälouma and Harting (2008) further proposed the use of a special surface patterning which can produce positive slip in one direction but negative in another direction. Such a surface might be useful in controlling throughput in microfluidic devices.

The dramatic feature of possible negative slip over a bubble mattress motivated Davis and Lauga (2009) to develop an analytical model to account for this change from slip to friction as a function of the bubble protrusion angle. Their model is only for the 2D transverse flow over cylindrical bubbles in the dilute limit (i.e. for a very small fraction of area covered by bubbles). Yet, their model can yield results comparable to those for 3D flow over spherical bubbles, as numerically computed by Steinberger et al. (2007) and Hyvälouma and Harting (2008).

An analogous model for 2D longitudinal flow over bubbles in the dilute limit was recently derived by Crowdy (2010). A more detailed study of the longitudinal flow over a curved stress-free interface was conducted by Teo and Khoo (2010). Using a finite-element Poisson equation solver, these authors numerically computed Couette and Poiseuille flows past a superhydrophobic surface made up of longitudinal grooves and ribs. They found the importance of considering the curvature of the stress-free interface in determining the
effective slip length. They also compared the performance of longitudinal grooves with those corresponding to transverse grooves, for which they used the dilute limit developed by Davis and Lauga (2009).

As remarked by Steinberger et al (2007), controlling slippage at the wall using superhydrophobic surfaces is still an open problem for surface engineering. A pattern should be designed to minimize the meniscus curvature. It is important that more extensive results be made available to the microfluidics community to help quantify the effective slip length as a function of the meniscus curvature and other parameters. This will, among others, facilitate the interpretation of experimental data. The analytical dilute limits and the simulation studies mentioned above are pioneering, and they have already provided some important clues to the problems. However, there is still a lack of models that are applicable beyond the dilute limit and can generate results more readily.

The objective of this study is to present semi-analytic models and results for Stokes flow over a periodic array of protrusions on an otherwise no-slip plane surface. Particular attention is paid to 2D bumps, which are ridges (if non-slipping) or cylindrical bubbles (if perfect-slipping), and 3D bumps, which are round bosses (if non-slipping) or spherical bubbles (if perfect-slipping). Being applicable to any finite fractions of area covered by the bumps, our models are used to check the range of validity of the dilute limits. Extensive results are presented in this paper showing the basic trends for the dependence of the effective slip length on the flow direction, protrusion angle, area coverage and slip condition of the bump surface. Our results are also compared with those simulations performed by previous authors. The results presented in this paper are more comprehensive than those available in the literature. We shall also derive phenomenological equations, by curve-fitting the numerical data, for the effective slip length as a function of the area fraction for some optimum or extremum values of the protrusion angle.

By Navier’s slip condition (Navier 1823), the slip length is defined to be the depth in a reference surface at which the velocity profile would extrapolate to zero. In the present problem, two types of slip are considered: the intrinsic (or microscopic) slip and the effective (or macroscopic) slip. Intrinsic slip can be due to chemical treatment of a surface, which becomes hydrophobic, and an aqueous liquid will flow past the surface with partial slip. The slip will become infinite when the surface is a no-shear boundary of a perfect-slipping medium (e.g. inviscid gas). The reference surface for the intrinsic slip length is the material surface itself. The effective slip is essentially an averaged boundary slip for flow past a heterogeneous surface. For the present patterned surfaces, the reference surface for the effective slip length is located on the flat part of the boundary. See Lauga et al (2007) for some detailed descriptions of the boundary slip.

2. 2D cylindrical protrusions

We first consider 2D longitudinal and transverse flows of a fluid over a periodic array of cylindrical bumps on an otherwise no-slip plane surface. The bumps are separated by a distance $2L$, and their width on the surface is $2aL$, where $0 < a < 1$ is the area fraction of the surface covered by bumps. Let us normalize lengths by $L$. Figure 1 shows a definition sketch of our normalized problem. The bumps are geometrically a circular arc defined by the half-width, $a$, and the protrusion angle, $-\pi/2 \leq \theta \leq \pi/2$. The circular arc is given by $y = \chi(x) \equiv \pm (R^2 - x^2)^{1/2} \mp (R^2 - a^2)^{1/2}$ for $|x| < a$, where the upper/lower signs are for positive/negative protrusion angles, and $R$ is the radius of curvature given by $a = R \sin|\theta|$. A negative protrusion angle will turn bumps into troughs on the plane surface. For bumps or troughs, the peak elevation or depression of the pattern is equal to $R - (R^2 - a^2)^{1/2}$. 
Figure 1. (a) Longitudinal and transverse flows over a periodic array of cylindrical bumps. The flow is driven by a unity velocity gradient in the far field. (b) The bump is a circular arc defined by the half-width $a$ and the protrusion angle $\theta$. The radius of curvature of the arc is $R$, and the curved surface has a partial slip length $\lambda$. The lengths are normalized by half the period of the pattern. The area fraction of the surface covered by bumps is given by $\alpha$.

The curved surface is assumed to have an intrinsic partial-slip length $\lambda$ that can vary in the range $0 \leq \lambda \leq \infty$. The lower limit $\lambda = 0$ means a no-slip bump surface, while the upper limit $\lambda = \infty$ corresponds to a stress-free condition. The latter is a good assumption when the curved surface is a meniscus between a viscous liquid and a gas bubble, where the surface curvature results from a balance of forces due to pressure difference and surface tension. The bubbles can maintain a circular shape when the surface tension is so strong compared with the applied shear rate that the Capillary number, which is the ratio of viscous to surface forces, is much less than unity (say, of the order of 0.01) (Hyväluoma and Harting 2008). We also assume that, owing to a very small length scale, the Reynolds number is so small that the inertia can
be ignored. In what follows, Stokes flow is assumed, and the velocities are normalized by \((L \tau/\mu)\), where \(\tau\) is the shear stress at \(y = \infty\) and \(\mu\) is the fluid viscosity.

### 2.1. Longitudinal flow

For Stokes flow that is parallel to the surface pattern and is driven by a unity velocity gradient far above the surface, the longitudinal velocity is (Ng and Wang 2009)

\[
w(x, y) = \delta_\parallel + y + \sum_{n=1}^{\infty} A_n \cos(\alpha_n x) e^{-\alpha_n y},
\]

where \(\alpha_n = n\pi\), \(\delta_\parallel\) is the effective longitudinal slip length and \(A_n\) are unknown coefficients.

Note that the flow is symmetrical about the plane \(x = 0\); \(w\) is even in \(x\).

On the curved surface, \(0 \leq x < a\), \(y = y_c(x)\) the unit outward normal vector is \(\vec{n} = (x, y'/R)\), and hence the partial-slip condition can be found to be

\[
w \mp \lambda R \left( x \frac{\partial w}{\partial x} + y' \frac{\partial w}{\partial y} \right) = 0,
\]

where \(y' = \pm(R^2 - x^2)^{1/2}\), and the upper/lower signs are for positive/negative protrusion angles. On the plane surface, \(a < x \leq 1\), \(y = 0\), the no-slip condition is satisfied: \(w(x, 0) = 0\).

Applying the no-slip condition at the point \((1,0)\) in particular, we obtain an expression for the slip length

\[
\delta_\parallel = -\sum_{n=1}^{M} A_n \cos(\alpha_n),
\]

where we have truncated \(A_n\) to \(M\) terms. The \(M\) unknown coefficients, \(A_1, ..., M\), are determined by prescribing the mixed boundary conditions on the surface in an approximate sense. For positive protrusion angles, the point collocation method works well. The boundary conditions are to be satisfied at \(M\) discrete points, which are evenly distributed on the curved part and the plane part of the surface but avoiding the corner point at \(x = a\). For negative protrusion angles, the corner singularity becomes stronger, and the method of subdomain is used instead. The boundary surface is divided into \(M\) subdomains of equal length, over each of which the boundary condition is integrated. Hence, the partial-slip or no-slip conditions are satisfied in an integral sense in each subdomain. By either method, we can form a system of \(M\) linear equations for the unknowns, which can be solved with a standard solver. Typically, \(M \sim 50-70\) is good enough to yield results with an order of accuracy \(10^{-3}\), which is consistent with that in previous studies using similar techniques (e.g. Wang 2003, Ng and Wang 2009).

Figure 2 shows comparisons of our results for \(\delta_\parallel^\infty\), where the superscript denotes \(\lambda = \infty\), with the finite-element results of Teo and Khoo (2010), the lattice Boltzmann model (LBM) simulation results of Hyväluoma and Harting (2008) and the analytical dilute limits of Crowdy (2010).

Employing a commercial finite-element solver, Teo and Khoo (2010) numerically computed flow past a superhydrophobic surface with longitudinal grooves taking into account the effects due to curvature of the liquid–gas interface. They used some 400,000 triangular elements to discretize their computational domain. Our semi-analytic method demands much less computational effort. Figure 2(a) shows our results to be in excellent agreement with those presented in figure 4 of Teo and Khoo (2010). This lends support to the numerical accuracy of our results.
For the longitudinal flow over 2D bubbles, Crowdy (2010) deduced the following analytical dilute limit for $\delta_\infty$: \[
\delta_\infty(\theta, a \ll 1) = \frac{\pi a^2}{12} \left[ \frac{3\pi^2 - 4\pi \theta + 2\theta^2}{(\pi - \theta)^2} \right]. \tag{4}
\]

We show in figures 2(a) and (b) how our results compare with this analytical limit. For any protrusion angle $\theta$ and stress-free area fraction $a$, the analytical limit underpredicts the...
Figure 3. Longitudinal slip length $\delta_\parallel$ as a function of the protrusion angle $\theta$ and the area fraction $a$, where (a) $\lambda = \infty$, (b) $\lambda = 10$, (c) $\lambda = 1$ and (d) $\lambda = 0$. The solid circles in panel (a) are values computed by the analytical formula (5).

The percentage difference is, however, relatively mild unless $a > 0.5$, whereas for fixed $a$ the difference increases monotonically with increasing $\theta$. This comparison suggests that one can safely use the dilute limit (within a 10% difference) for $a < 0.35$ for all protrusion angles.

Further, we compare in figure 2(c) our results with the LBM simulation results as presented in figure 4 of Hyväläuoma and Harting (2008), for a particular protrusion angle $\theta = 72^\circ$. For the longitudinal slip length, the agreement in prediction between the two approaches is very good, despite some unrealistic parameters being used in the LBM simulations (see further comparisons with their results below).

In figure 3, we further show the effective longitudinal slip length $\bar{\delta}_\parallel^\lambda(\theta, a)$ as a function of the protrusion angle $\theta$, the area fraction $a$ and the partial slip length $\lambda$. The case $\lambda = \infty$ shown in figure 3(a) has already been discussed in detail by Teo and Khoo (2010). The effective slip length remains nearly constant when the protrusion is negative. For positive protrusion, the effective slip length increases with an increasing rate to reach the maximum as the angle increases to approach $90^\circ$. The asymmetry in the slip length between concave and convex bubbles has already been noted in the literature. In a previous study, the authors Ng and Wang (2009) attempted to model liquid penetration into grooves assuming a flat interface. Teo and Khoo (2010) found that this assumption can lead to underprediction of the effective slip length.
Figure 4. Polynomial curves fitting the numerical data (squares) for $\delta_1(\theta = 90^\circ, a)$ and $\delta_0(\theta = 90^\circ, a)$. The dashed line is the dilute limit given by equation (8).

For the limiting case of flat shear-free longitudinal slots, $\theta = 0^\circ$, our results agree very well with the formula of Philip (1972):

$$\delta_1(\theta = 0^\circ, a) = \frac{2}{\pi} \ln \left( \frac{\sec \left( \frac{\pi a}{2} \right)}{2} \right). \tag{5}$$

The results obtained by this analytical formula are shown by the solid circles in figure 3(a).

Because of gas viscosity or surface impurities (e.g. the presence of surfactants), perfect slip over the bubbles may not be achieved, and a large partial slip can be considered instead. Figure 3(b) shows that a large but finite value of $\lambda = 10$ will not materially affect the effective slip length (compared with the $\lambda = \infty$ limit), except when $a \geq 0.7$, especially for $\theta > 60^\circ$. Clearly, a wider bubble at a larger protrusion angle is more susceptible to a reduction of the partial slip length. When $\lambda = 1$, the trend is even reversed for $a \geq 0.7$; the slip length decreases with increasing $\theta$. In the no-slip limit $\lambda = 0$, the effective slip length turns negative for any positive protrusion. This is expected since no-slip bumps on the surface are surface roughness that can only add resistance to the flow. The negative slip length will increase in magnitude as $\theta$ increases towards $90^\circ$, and is larger in magnitude for larger $a$.

It is of particular interest whether some convenient-to-use formulae can be made available for the evaluation of the effective slip length for two limiting cases: perfect-slip and no-slip semi-circular bumps. Figure 4 shows our computed data (squares) together with the polynomial curves (solid lines) best-fitting the data. The polynomials, which need to be of some high order to give satisfactory data fitting in the range $0 \leq a \leq 0.9$, are found to be

$$\delta_1(\theta = 90^\circ) = 86.56a^6 - 173.88a^5 + 134.28a^4 - 46.74a^3 + 8.93a^2 - 0.40a, \tag{6}$$

$$\delta_0(\theta = 90^\circ) = 0.85a^3 - 1.72a^2 - 0.005a, \tag{7}$$

where the superscript denotes the value of $\lambda$. In this figure, we show also the dilute limit for $\delta_1(\theta = 90^\circ)$, as can be deduced by the theory of Sarkar and Prosperetti (1995):

$$\delta_1(\theta = 90^\circ, a \ll 1) = -\frac{\pi}{2} a^2. \tag{8}$$

We find that this dilute limit, which predicts a higher magnitude, is within a 10% difference from our results as long as $a < 0.35$.  

8
2.2. Transverse flow

For Stokes flow that is perpendicular to the surface pattern and is driven by a unity velocity gradient far above the surface, the \( x \)- and \( y \)-components of velocity are (Ng and Wang 2009)

\[
\begin{align*}
    u(x, y) &= \delta_\perp + y + \sum_{n=1}^{\infty} \left[ -B_n + C_n \left( 1 - \alpha_n y \right) \cos(\alpha_n x) e^{-\alpha_n y} \right], \\
    v(x, y) &= \sum_{n=1}^{\infty} \left[ B_n + C_n \alpha_n y \right] \sin(\alpha_n x) e^{-\alpha_n y},
\end{align*}
\]

where \( \alpha_n = n\pi \), \( \delta_\perp \) is the effective transverse slip length and \( B_n \) and \( C_n \) are unknown coefficients. Note that \( u \) is even in \( x \), while \( v \) is odd in \( x \).

On the curved surface, \( 0 \leq x < a \), \( y = y_c(x) \), the unit normal and tangent vectors are \( \vec{n} = (x, y')/R \) and \( \vec{t} = (-y', x)/R \), by which the conditions of partial-slip and zero normal velocity are found to be, respectively,

\[
\begin{align*}
    y' u - x v \pm \lambda R \left[ 2xy' \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) + \left( x^2 - y'^2 \right) \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] &= 0, \\
    xu + y'v &= 0,
\end{align*}
\]

where \( y' = \pm (R^2 - x^2)^{1/2} \), and the upper/lower signs are for positive/negative protrusion angles. On the plane surface, \( a < x \leq 1 \), \( y = 0 \), both velocity components are zero: \( u(x, 0) = 0 \), \( v(x, 0) = 0 \).

Applying the no-slip condition at the point \((1, 0)\) in particular, we obtain an expression for the effective slip length

\[
\delta_\perp = \sum_{n=1}^{M} \left( B_n - C_n \right) \cos(\alpha_n),
\]

where we have truncated \( B_n \) and \( C_n \) each to \( M \) terms. The \( 2M \) unknown coefficients are determined using similar numerical methods to those described above for the longitudinal flow. For positive protrusion angles, the two pairs of boundary conditions are enforced at \( M \) discrete points evenly distributed on the boundary surface. For negative protrusion angles, the boundary conditions are integrated in \( M \) subdomains evenly dividing the boundary surface.

For transverse flow over a 2D bubble mattress, the following dilute limit was developed by Davis and Lauga (2009):

\[
\delta^\infty_\perp (\theta, a \ll 1) = \pi a^2 \int_0^\infty A(\theta, s) \, ds,
\]

where the superscript \( \infty \) denotes \( \lambda = \infty \), and

\[
A(\theta, s) = \frac{s}{\sinh[2s(\pi - \theta)] + s \sin(2\theta)} \times \left[ \frac{\cos(2\theta) + s \sin(2\theta) \cosh(s\pi) + \sinh(s(\pi - 2\theta))}{\sinh(s\pi)} \right].
\]

This analytical dilute limit is compared with our results, as shown in figures 5(a) and (b). The basic features displayed by the transverse effective slip length as a function of the protrusion angle were noted previously by Davis and Lauga (2009). Again, there is asymmetry in the slip length between positive and negative protrusion angles. The effective slip length increases...
with increasing $\theta$ until it reaches a maximum at $\theta \approx 10^\circ$. Let us call this angle the optimum angle. The slip length then decreases with further increase in $\theta$. Eventually, the slip length becomes zero at $\theta \approx 65^\circ$, which is called the critical angle $\theta_c$ by Davis and Lauga (2009). This angle marks a geometric transition from reduced to enhanced friction of the bubble mattress. The slip length drops to become negative when the critical angle is exceeded. Figures 5(a) and (b) reveal that the dilute model given in equation (14) predicts a lower magnitude for $\theta < \theta_c$, but a larger magnitude for $\theta > \theta_c$, where the deviation varies with $\theta$ and $a$. Again, our comparison suggests that it would be safe to use the dilute model for $a < 0.35$ for an accuracy within 10%. Near the two angles ($\theta \approx 10^\circ$, $\theta = 90^\circ$) at which the slip length is maximum in magnitude, the dilute model prediction can appreciably deviate from the real value when $a > 0.5$. One can picture that when the slip length is maximum in magnitude, the no-slip plane is farthest apart from the reference surface.

It is interesting to note that the critical angle is practically the same (i.e. $\theta_c \approx 65^\circ$) independent of $a$ and therefore the dilute model given by equation (14) gives correct
predictions for a protrusion angle near the critical angle even when the sparse condition ($a \ll 1$) is far exceeded. When the protrusion angle is at the critical value, the bump surface will have zero effective slippage. It means that the flow over a slipping bump at the critical protrusion angle is macroscopically not different from that over a no-slip flat surface. This remains true whether individual bumps are sparsely separated or not. This explains why the critical protrusion angle is not sensitive to $a$.

Further, we compare in figure 5(c) our results with the LBM simulation results as presented in figure 4 of Hyvälouma and Harting (2008), for a particular protrusion angle $\theta = 72^\circ$. For the transverse slip length, the LBM results are about 20–60% higher in magnitude than our results, although the qualitative trend of an increasingly negative slip length with increasing $a$ is similarly predicted by both the approaches. As remarked by Hyvälouma and Harting (2008) themselves, they chose in their simulations a density ratio of 22 between a liquid and a gas. This ratio is too small for a realistic description of a gas interacting with a liquid. Furthermore, the multiphase LBM they used is a diffuse interface model, and the protrusion angle is not defined uniquely but depends on where the actual interface location is assumed. A finite width of the interface implies a finite slip due to friction within the interface region. These deficiencies may affect the quantitative accuracy of the LBM results. Hyvälouma and Harting (2008) could only put emphasis on the qualitative insight obtained from their simulations.

In figure 6, we further show the effective transverse slip length $\delta^\perp(\theta, a)$ as a function of the protrusion angle $\theta$, the area fraction $a$ and the partial slip length $\lambda$. A reduction of the partial slip length from $\lambda = \infty$ to $\lambda = 10$ will only modestly change the effective slip length. A further reduction of the partial slip length to $\lambda = 1$ will see more appreciable friction effects, especially near the two protrusion angles $\theta \approx 10^\circ$, $\theta = 90^\circ$ at which the effective slip length is maximum in magnitude (i.e. when the no-slip plane is farthest away from the reference surface). It is interesting to find that both the critical and optimum angles (at which the effective slip length is, respectively, zero and the maximum positive) decrease with decreasing $\lambda$, until they reach $\theta = 0^\circ$, $-90^\circ$ when $\lambda = 0$. Comparing figures 3 and 6, one can find $\delta_1$ and $\delta_\perp$ are very differently affected by the protrusion angle, qualitatively and quantitatively, when $\lambda > 0$. However, when $\lambda = 0$, they exhibit nearly the same dependence on the protrusion angle. In other words, the flow direction will have less influence on the effective slip (or effective friction) when the bumps are non-slipping than when they are perfect-slipping.

In the limiting case of flat shear-free transverse slots, $\theta = 0^\circ$, our results agree very well with the formula obtainable from Philip (1972):

$$\delta^\perp(\theta = 0^\circ, a) = \frac{1}{\pi} \ln \left[ \sec \left( \frac{\pi a}{2} \right) \right]. \quad (16)$$

The results obtained by this analytical formula are shown by solid circles in figure 6(a).

Let us also provide some formulae by which the effective slip length for some particular cases can be evaluated readily. For the two limiting cases of perfect-slip and no-slip semi-circular bumps and for the peak value at the optimum angle, figure 7 shows our computed data (squares) together with the polynomial curves (solid lines) best-fitting the data. The polynomials, which fit the data in the range $0 \leq a \leq 0.9$, are found to be

$$\delta^\perp(\theta = 10^\circ) = 3.11a^5 - 4.72a^4 + 2.81a^3 - 0.24a^2 + 0.05a, \quad (17)$$

$$\delta^\perp(\theta = 90^\circ) = 0.43a^3 - 0.86a^2 - 0.003a, \quad (18)$$

$$\delta^\perp(\theta = 90^\circ) = -1.35a^4 + 3.30a^3 - 2.87a^2 - 0.01a, \quad (19)$$
Figure 6. Transverse slip length $\delta_\perp$ as a function of the protrusion angle $\theta$ and the area fraction $a$, where (a) $\lambda = \infty$, (b) $\lambda = 10$, (c) $\lambda = 1$ and (d) $\lambda = 0$. The solid circles in panel (a) are values computed by the analytical formula (16).

Figure 7. Polynomial curves fitting the numerical data (squares) for $\delta_\perp(\theta = 10^\circ, a)$, $\delta_\perp(\theta = 90^\circ, a)$ and $\delta_\parallel(\theta = 90^\circ, a)$. The dashed line is the dilute limit given by equation (20).

where the superscript denotes the value of $\lambda$. In figure 7, we also show the dilute limit for $\delta_\perp(\theta = 90^\circ)$, which is deduced by us following the models of Davis and O’Neill (1977) and Davis and Lauga (2009):

$$\delta_\perp(\theta, a \ll 1) = \pi a^2 \int_0^\infty B(\theta, s) \, ds,$$

(20)
where

\[ B(\theta, s) = \frac{s}{s^2 \sin^2(\theta) - \sinh^2[s(\pi - \theta)]} \]
\[ \times \left[ s \sin(\theta) \cos(\theta) + \frac{s^2 \sin^2(\theta) \cosh(s\pi) + \sinh(s\theta) \sinh[s(\pi - \theta)]}{\sinh(s\pi)} \right]. \tag{21} \]

The applicability of this no-slip dilute limit is found to be more restricted than its perfect-slip counterpart (equation (14)). It can only be used for \( a < 0.2 \) for an accuracy within 10%, and for higher \( a \) the percentage error can be quite large.

3. 3D spherical protrusions

We next consider 3D flow over a square array of spherical bumps on an otherwise no-slip plane surface. The bumps are separated by a distance \( 2L \) in the streamwise and spanwise directions, and the bump radius on the surface is \( aL \), where \( 0 < a < 1 \). The area fraction of the surface covered by bumps is \( \phi = \pi a^2/4 \). Let us normalize all length quantities by \( L \). Figure 8 shows a definition sketch of our normalized problem for the 3D flow over the protrusions. For simplicity, we shall consider only positive protrusion in this problem. Also, only the limiting perfect-slip and no-slip conditions are assumed for the bump surface.

The bumps are geometrically a round cap defined by the bump radius, \( a \), and the protrusion angle, \( 0 < \theta \leq \pi/2 \). The round cap is given by the surface \( z = z_c(x, y) \equiv (R^2 - x^2 - y^2)^{1/2} = (R^2 - a^2)^{1/2} \) for \( x^2 + y^2 < a^2 \), where \( R \) is the normalized radius of curvature given by \( a = R \sin \theta \). The peak protuberance of the bumps is equal to \( R - (R^2 - a^2)^{1/2} \).

For Stokes flow driven by a unity velocity gradient in the \( x \)-direction far above the surface, the \( x \)-, \( y \)-, and \( z \)-components of velocity are available on adopting the analysis of Ng and Wang (2010):

\[
\begin{align*}
    u(x, y, z) &= z + \delta_b + \sum_{n=1}^{\infty} \left[ A_{1n} + A_{2n} \right] \cos(\alpha_n x) e^{-\alpha_n z} + \sum_{m=1}^{\infty} A_{3m} \cos(\alpha_m y) e^{-\alpha_m z} \\
    &\quad + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ B_{1nm} + B_{2nm} z \right] \cos(\alpha_n x) \cos(\alpha_m y) e^{-\gamma_{nm} z}, \tag{22}
\end{align*}
\]

\[
\begin{align*}
    v(x, y, z) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ B_{1nm} - \frac{\alpha_m}{\alpha_n} B_{2nm} z \right] \sin(\alpha_n x) \sin(\alpha_m y) e^{-\gamma_{nm} z}, \tag{23}
\end{align*}
\]

\[
\begin{align*}
    w(x, y, z) &= -\sum_{n=1}^{\infty} \left[ A_{1n} + A_{2n} \left( z + \frac{1}{\alpha_n} \right) \right] \sin(\alpha_n x) e^{-\alpha_n z} \\
    &\quad - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ \frac{\alpha_n}{\gamma_{nm}} B_{1nm} = \frac{\alpha_m}{\gamma_{nm}} B_{3nm} + B_{2nm} \left( \frac{\gamma_{nm}}{\alpha_n} z + \frac{1}{\alpha_n} \right) \right] \\
    &\quad \times \sin(\alpha_n x) \cos(\alpha_m y) e^{-\gamma_{nm} z}, \tag{24}
\end{align*}
\]

where \( \alpha_n = n \pi \), \( \gamma_{nm} = (\alpha_n^2 + \alpha_m^2)^{1/2} \), \( \delta_b \) is the effective slip length and \( A_{1n} \), \( A_{2n} \), \( A_{3m} \), \( B_{1nm} \), \( B_{2nm} \) and \( B_{3nm} \) are coefficients yet to be determined. Note that \( u \) is even in both \( x \) and \( y \), \( v \) is odd in both \( x \) and \( y \), while \( w \) is odd in \( x \) but even in \( y \).
On the curved surface, \(0 \leq x^2 + y^2 < a^2, z = z_s(x, y)\), the unit outward normal vector is \(\hat{n} = (x, y, z')/R\), where \(z' = (R^2 - x^2 - y^2)^{1/2}\). We further choose two unit tangent vectors, \(\hat{t}_1 = (z', 0, -x)/(x^2 + z'^2)^{1/2}\) and \(\hat{t}_2 = (0, z', -y)/(y^2 + z'^2)^{1/2}\), which are tangent to the surface in the \((x, z)\) and \((y, z)\) planes, respectively. Based on these normal and tangent vectors, we can obtain the following conditions on the curved surface:

\[
2xz' \left( \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \right) + (z'^2 - x^2) \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\
+ yz' \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) - xy \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0 \tag{25}
\]
for stress-free along $\vec{t}_1$,
\[
2yz' \left( \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \right) + \left( z'^2 - y^2 \right) \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\
+ xz' \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - xy \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0
\]
(26)
for stress-free along $\vec{t}_2$ and
\[
 xu + yv + z'w = 0
\]
(27)
for zero normal velocity on the surface. If the bump surface is non-slipping, the boundary conditions are simplified to
\[
u(x, y, z_s) = 0, \quad w(x, y, z_s) = 0.
\]
On the plane surface, $x^2 + y^2 > a^2, x \leq 1, y \leq 1$ and $z = 0$, all the velocity components are zero by virtue of the no-slip and zero normal flux conditions.

On applying the zero-velocity conditions along the lines $y = 1$ and $x = 1$ in particular, $u(x, 1, 0) = 0, u(1, y, 0) = 0$ and $w(x, 1, 0) = 0$, we can further deduce that
\[
A_{1n} = - \sum_{m=1}^{M} B_{1nm} \cos(\alpha_m),
\]
(28)
\[
A_{2n} = \sum_{m=1}^{M} \left[ \alpha_n \left( 1 - \frac{\alpha_n}{\gamma_{nm}} \right) B_{1nm} - B_{2nm} + \frac{\alpha_n \alpha_m}{\gamma_{nm}} B_{3nm} \right] \cos(\alpha_m),
\]
(29)
\[
A_{3m} = - \sum_{n=1}^{M} B_{1nm} \cos(\alpha_n),
\]
(30)
\[
\delta_b = \sum_{n=1}^{M} \sum_{m=1}^{M} B_{1nm} \cos(\alpha_n) \cos(\alpha_m),
\]
(31)
where we have truncated $B_{1nm}, B_{2nm}$ and $B_{3nm}$ each to $M \times M$ terms. These $3M^2$ unknown coefficients are to be determined by applying the boundary conditions on the curved as well as the plane parts of the surface. Again, the method of point collocation is adopted. We enforce the boundary conditions to be satisfied at $M^2$ points evenly distributed on the bump and the plane surfaces. At each point, three conditions are applied. Therefore, a system of $3M^2$ linear equations is formed for the same number of unknowns and can be solved readily with a standard solver.

Figure 9 shows a comparison, for flow over perfect-slip bubbles, between the numerical data of two previous studies and our modeling results. These numerical data are adopted from the results presented in figure 3 of Steinberger et al (2007) and figure 1 of Hyvälouma and Harting (2008).

Using a commercial multiphysics computational package, Steinberger et al (2007) performed some 3D numerical simulations of a Couette flow between a smooth moving surface and a fixed surface made up of a square array of bubbles in an attempt to compare with their experiment. The same problem was later investigated in further detail, using LBM, by Hyvälouma and Harting (2008), who also considered the effect of the Capillary number. The methods of these previous authors are, however, rather computationally involved, and the effective slip length has to be inferred indirectly from the computed flow field. In contrast, our modeling is less computationally intensive and can yield the effective slip length more directly.
Hyvälouma and Harting (2008) remarked that the maximum slip length is obtained when the protrusion angle is equal to zero. This is not true. As in the case of transverse flow discussed above, the effective slip length here will reach the maximum at a positive protrusion angle. It is true, however, that the maximum slip length, which is obtained at a protrusion angle in the range of 10–20°, is only modestly larger than the slip length at the zero protrusion angle. This mild rise in slip length from $\theta = 0^\circ$ to $20^\circ$ is not captured in the simulation results in the previous two studies. We also find that the critical protrusion angle at which the slip length becomes zero is approximately 66°, which is close to the one for the 2D transverse flow reported above. The critical protrusion angle that can be inferred from the results of Steinberger et al. (2007) and Hyvälouma and Harting (2008) is approximately 62° and 69°, respectively.

In figure 10, we further show the effective slip length $\delta_b(\theta, \phi)$ as a function of the protrusion angle $\theta$ for the perfect-slip ($\lambda = \infty$) and no-slip ($\lambda = 0$) limits, where the area fraction of the surface covered by bumps $\phi = 0.3, 0.5$ and 0.7. The results for shear-free circular patches (i.e. zero protrusion angle) are checked to agree with those reported previously by us (Ng and Wang 2010). Comparing this figure with figure 6, one finds a strong similarity in the dependence of the effective slip length on the protrusion angle between the cases of 3D flow and 2D transverse flow over the bumps. The similarity is both quantitative and qualitative. Given the same fraction of surface area covered by bumps, the slip length varies with the protrusion angle in nearly the same manner (within a factor of two) for either of the two flow cases. This explains why Davis and Lauga (2009) could compare their 2D model with the 3D simulation results of Steinberger et al. (2007) and Hyvälouma and Harting (2008).

Let us also provide some formulae by which the effective slip length for the particular cases can be evaluated readily. For the two limiting cases of perfect-slip and no-slip hemispherical bumps ($\theta = 90^\circ$) and for flat circular shear-free patches ($\theta = 0^\circ$), figure 11 shows our computed data (squares) together with the polynomial curves (solid lines) best-fitting the data as functions of the bump surface coverage $\phi = \pi a^2 / 4$. The polynomials, which fit the data in the range $0 \leq \phi \leq 0.7$ (or $0 \leq a \leq 0.94$), are found to be

\begin{align}
    \delta_\infty^\infty(\theta = 0^\circ) &= 0.35\phi^3 + 0.14\phi^2 + 0.12\phi, \\
    \delta_\infty^\infty(\theta = 90^\circ) &= 0.45\phi^3 - 0.67\phi^2 - 0.21\phi, \\
    \delta_0^\infty(\theta = 90^\circ) &= 1.32\phi^3 - 1.40\phi^2 - 0.65\phi, \\
\end{align}
Figure 10. Effective slip length $\delta_b$ as a function of the protrusion angle $\theta$ for the bump surface coverage $\phi = \pi a^2/4 = 0.3, 0.5$ and 0.7, where (a) $\lambda = \infty$, (b) $\lambda = 0$.

Figure 11. Polynomial curves fitting the numerical data (squares) for $\delta_b^\infty(\theta = 0^\circ, \phi)$, $\delta_b^\infty(\theta = 90^\circ, \phi)$ and $\delta_b^0(\theta = 90^\circ, \phi)$. The dashed lines are the dilute limits given by equations (35) and (36). The open circles are values computed by formula (37).
where the superscript denotes the value of $\lambda$. In figure 11, we also show the dilute limits (dashed lines) for the cases

$$\delta_\infty^b (\theta = 0^\circ, \ a \ll 1) = \frac{2}{5} a^3, \quad (35)$$

which is given by Sbragaglia and Prosperetti (2007a), and

$$\delta_\infty^b (\theta = 90^\circ, \ a \ll 1) = -\frac{\pi}{6} a^3, \quad (36)$$

which is given by Davis and Lauga (2009). Also shown in figure 11 for comparison are values (open circles) computed by the polynomial formula deduced heuristically (in an attempt to extend their analytical expression beyond the dilute limit) by Sarkar and Prosperetti (1995):

$$\delta_0^b (\theta = 90^\circ) = -(3 - 2.73\phi + 0.96\phi^2) \frac{\pi}{6} a^3. \quad (37)$$

Note that the slip length given in equation (36) also corresponds to the average normalized height of the interface. This dilute limit for perfect-slip hemispherical bumps gives a satisfactory prediction (within a 10% difference from the true value) as long as $\phi < 0.3$. The dilute limit (35) for perfect-slip circular patches predicts well over a larger range of parameters, $\phi < 0.4$. For no-slip hemispherical bumps, our results agree very well with the predictions by the extended formula (37) of Sarkar and Prosperetti (1995).

4. Concluding remarks

We have developed semi-analytic models, based on the methods of eigenfunction expansions and point collocation, for 2D or 3D Stokes flows over a surface patterned with cylindrical or spherical bumps. Our models can generate the results more readily than fully numerical (such as finite-element or lattice Boltzmann) models for the effective slip length.

The analytical dilute limits are found to predict well for an area fraction not exceeding 0.1–0.35, depending on the protrusion angle, flow direction and the slip condition of the protruding surface. One is cautioned that the error can be quite large if a dilute limit is used for an area fraction larger than 0.5 and a protrusion angle near the extremum, i.e. $90^\circ$.

Owing to gas viscosity or surface impurities, a bubble surface may not be perfectly slipping. We have investigated for the 2D case and found that a large but finite partial slip on the protruding surface will not materially lower the effective slip in the range of parameters considered. The effect is significant only when the flow is longitudinal, where the protrusion area fraction is close to unity, and the protrusion angle is nearly $90^\circ$.

For longitudinal flow over cylindrical bubbles, the effective slip length increases monotonically with the protrusion angle. For transverse flow and for 3D flow over spherical bubbles, the effective slip length reaches a positive maximum at an optimum protrusion angle in the range of 10–20°, becomes zero at the critical protrusion angle of approximately 65° and drops to the negative maximum at the extremum protrusion angle of 90°. This trend is insensitive to changes in the area fraction of the protrusions. The effective roughness/slip over a surface with 3D protrusions is comparable to that over a surface with 2D transverse protrusions with the same protrusion angle and area fraction.

We have also derived by polynomial curve fitting some phenomenological equations for the effective slip length as a function of the protruding area fraction for some particular values
of the protrusion angle. These equations will enable quick evaluations and should be of value to those who wish to design a microchannel with a patterned wall.

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