USE OF MATHCAD AS A CALCULATION TOOL FOR WATER WAVES OVER A STRATIFIED MUDDY BED

CHIU-ON NG* and HOK-SHUN CHIU†
Department of Mechanical Engineering,
The University of Hong Kong,
Pokfulam Road, Hong Kong, China
Tel: (852) 2859 2622; Fax: (852) 2858 5415
*cong@hku.hk
†h0430909@hkusua.hku.hk

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This note marks the debut of a Mathcad worksheet, which has been developed to aid engineers in the calculation of properties of a surface wave propagating in water over a two-layer viscoelastic muddy bed. We first describe the problem formulation and then features of the worksheet. It is a very easy-to-use calculation tool. With the input of some basic parameters, such as the wave period and the fluid properties, one may get almost instantly the key results (wavenumber, wave damping rate, velocity and pressure fields, and so on) upon pressing a key. The worksheet has been extensively tested to ensure that it can produce reliable and accurate results.

Keywords: Water waves; stratified muddy bed; viscoelastic mud; wave attenuation; Mathcad.

1. Introduction

Although it is increasingly recognized that bed-layering is important to the modeling of wave-mud interaction, there virtually exists no handy tool by which engineers can do some quick calculations to evaluate the properties (wavelength, period, damping rate, flow and pressure fields) of surface water waves over a stratified muddy bed.

*Corresponding author.
The nonlinear fluidization model by Foda et al. [1993] is sophisticated, but rather algebraically and computationally intensive. It is desirable if a more ready-to-use calculation tool can be made available to aid engineers in this respect. The demand for such a calculation tool can only be increasing, since coastal development has now become a crucial part of the economy worldwide. This forms the basic motivation of the present work. Here, we introduce a worksheet that is built upon the software package, Mathcad, which offers an extremely convenient interface for the development as well as the use of such a calculation tool.

In this work, a system consisting of a water column overlying a two-layer muddy bed is considered. For engineering design purposes, it would suffice in most cases to divide the bed-mud into two discrete layers: the top layer represents the fluidized mud, while the bottom layer represents the settled bed. A two-layer bed model is deemed to be able to reveal much of the physics without excessively complicating the problem. The water and the two mud layers, each of which is assumed to be homogeneous, are separated by sharp interfaces without mixing. The mud is assumed to be a viscoelastic medium modeled as a Voigt body, which behaves like a viscous dashpot and an elastic spring operating in parallel. Under simple harmonic motions, the rheology can be conveniently represented by a complex viscoelastic parameter, in which the real part stands for the viscosity and the imaginary part stands for the elasticity. In this regard, various cases of the mud rheology can be considered within the scope of the model; the mud will be purely viscous, purely elastic, or viscoelastic when the viscoelastic parameter is real, imaginary, or complex in general, respectively.

The multi-layer mud model of Maa and Mehta [1990] is essentially an extension of the single-layer bed model due to Dalrymple and Liu [1978]. Our two-layer bed model is also based on the work of Dalrymple and Liu [1978], which is applicable when the wave amplitude is so small that the equations of motion can be linearized. As was remarked by Maa and Mehta [1990], the linear theory would work well under conditions wherein large bed deformations are not involved, as in a mild wave environment. The solutions to the linearized equations are analytical expressions in terms of transcendental functions of the layer thicknesses. For a two-layer bed model, there are altogether 15 complex unknowns in the whole set of solutions, including the two interfacial displacements and the wavenumber. These unknowns are to be determined using the kinematic and dynamic boundary conditions along the free water surface, the two interfaces and the bottom. Except the wavenumber which is an eigenvalue, the remaining unknowns are coefficients simply governed by a linear system of equations. Dalrymple and Liu [1978] used a complex secant method to iteratively solve the system of equations involving the search for the complex eigenvalue. Maa [1986] and Maa and Mehta [1990] further applied the secant method to finding solutions for a system involving any number of mud layers.

To alleviate the programming work, as well as to make the program easy to use, we here propose an alternative approach of solving the problem, namely by means of
the computing package, Mathcad. In the following sections, we shall further define the present problem, and briefly review the mathematical formulation and some basic expressions for the key variables. We then introduce a Mathcad worksheet, and explain how it can be utilized to determine the basic variables of interest. We then describe how the worksheet is verified for its numerical accuracy by comparing with available results in the literature. In particular, we show that our two-layer bed model can generate results that compare favorably with those generated by a four-layer bed model, as was presented by Maa and Mehta [1990]. Our Mathcad worksheet is useful to engineers and researchers as well.

2. Problem Formulation

Our problem formulation follows that of Maa and Mehta [1990]. Only an outline is given here. As is depicted in Fig. 1, we consider a system consisting of three layers of constant depths: a water column of depth \(d_w\), overlying two mud layers of depths \(d_1\) and \(d_2\). The subscripts \(w\), \(1\), and \(2\) are used to denote respectively the water layer, the upper and the lower mud layers. A stable stratification structure is assumed: the density profile is such that \(\rho_w < \rho_1 < \rho_2\). Water is a Newtonian fluid with viscosity \(\nu_w\), which can be the molecular or eddy viscosity depending on whether the flow is laminar or turbulent. For simplicity, a constant eddy viscosity is assumed. The mud rheology is characterized by a complex viscoelastic parameter

\[
\nu_j^\prime = \nu_j + iG_j/\rho_j\sigma \quad (j = 1, 2)
\]

![Fig. 1. Schematic diagram for waves in water over a two-layer muddy bed system.](image-url)
where $i$ is the complex unit, $\nu$ is the kinematic viscosity, $G$ is the shear modulus, and $\sigma = 2\pi/T$ and $T$ are respectively the frequency and period of the simple harmonic motion undergone by the mud.

Axes $x$ and $y$ are defined to be directed along the wave propagation, and vertically upward from the mean water level, respectively. There is a progressive wave propagating on the water surface, with an elevation given by

$$\eta = a \exp[i(kx - \sigma t)]$$

where $a$ is the surface wave amplitude at the origin $x = 0$, $k$ is the wavenumber, $\sigma$ is the wave frequency, and $t$ is time. Here, $a$ and $\sigma$ are assumed to be known real constants, while $k$ is a complex eigenvalue to be determined in the problem.

Given that the wave steepness $|ka| \ll 1$ is very small, the momentum equations can be linearized by ignoring the inertia terms as a first approximation. Solutions to the linear problem are hence expressible by an amplitude function of $y$ times the exponential factor:

$$\hat{f}_j(x, y, t) = f_j(y) \exp[i(kx - \sigma t)] \quad (j = w, 1, 2)$$

where $f$ may stand for the velocity components $(u, v)$ in the $x$- and $y$-directions, and the dynamic pressure $P$. The displacements on the upper and the lower interfaces take a similar form

$$\xi_j(x, t) = b_j \exp[i(kx - \sigma t)] \quad (j = 1, 2)$$

where $b$ is the interfacial wave amplitude.

On substituting Eq. (3) for $(\hat{u}, \hat{v})$ and $\hat{P}$ into the continuity and linearized horizontal momentum equations, we get

$$u_j = iv'_j/k \quad (j = w, 1, 2)$$

$$P_j = (\rho_j \nu'_j/k^2)[v''_j - v'_j \lambda^2_j] \quad (j = w, 1, 2)$$

where the prime indicates differentiation with respect to $y$, $\nu'_w = \nu_w$, and

$$\lambda^2_j = k^2 - i\sigma/\nu'_j \quad (j = w, 1, 2)$$

Further substituting these expressions into the vertical momentum equation, we can solve for the vertical velocities as

$$v_w(y) = A \sinh k(y + d_w) + B \cosh k(y + d_w)$$

$$+ C \exp(\lambda_w y) + D \exp[-\lambda_w(y + d_w)]$$

$$v_1(y) = E \sinh k(y + d_w + d_1) + F \cosh k(y + d_w + d_1)$$

$$+ G \exp[\lambda_1(y + d_w)] + H \exp[-\lambda_1(y + d_w + d_1)]$$

$$v_2(y) = I \sinh k(y + d_w + d_1 + d_2) + J \cosh k(y + d_w + d_1 + d_2)$$

$$+ M \exp[\lambda_2(y + d_w + d_1)] + N \exp[-\lambda_2(y + d_w + d_1 + d_2)]$$

(10)
where $A, B, \ldots, M$ and $N$ are twelve complex coefficients yet to be determined. We remark that in these expressions the hyperbolic terms containing $k$, and the exponential terms containing $\lambda_j$, constitute respectively the inviscid and viscous parts of the solution. In Eq. (8), the exponential terms associated with $C$ and $D$ are appreciable only near the boundary layers below the water surface and above the water-upper mud interface, respectively. This is because the water viscosity is so small that the viscous effect is significant in water only within these boundary layers, which are much thinner than the water depth. In contrast, the mud viscosity is typically so large that its effect can be significant across the entire mud layer. The thickness of the wave boundary layer, also known as the Stokes layer, is given by $\delta = (2\nu/\sigma)^{1/2}$. In water, $\delta_w \ll d_w$, while in the muds, $\delta_j \sim d_j$ ($j = 1, 2$). Therefore, for generality, all the exponential terms in Eqs. (9) and (10) are evaluated anywhere in the mud layers. By virtue of these assumptions, the present model is valid for any values of the mud depths, which can be large, small or even zero.

The twelve coefficients, together with the two interfacial wave amplitudes and the wavenumber, are to be determined using the kinematic and dynamic boundary conditions, as well as the continuity of stress and velocity components, on the free surface, on the two interfaces, and on the solid bottom. Details of these boundary and matching conditions are given in Dalrymple and Liu [1978] and Maa [1986]. Our approach is to first solve by Gauss elimination the linear set of equations for the twelve coefficients so that each of them can be explicitly expressed in terms of others symbolically. Then, the wavenumber is determined as an eigenvalue satisfying one of the dynamic boundary conditions on the free surface. The two interfacial wave amplitudes are then found from the corresponding kinematic boundary conditions. On this basis, the general purpose computational package, viz. Mathcad, is employed to perform the tasks of solution finding and numerical calculations. See Chiu [2007] for further details of the problem solving based on our Mathcad worksheet.

The Mathcad worksheet that we have developed is fully annotated, and is straightforward to use. After inputting values for the wave period, depths of the water and mud layers, and the fluid properties, results will be generated almost instantly upon pressing the function key $[F9]$. Between the inputs and the results are a large number of definitions and working equations, which require no action on the part of the user and should be left unchanged under all circumstances. To protect these equations and to make the document more readable, we have inserted an area, which is then set to be collapsed, to enclose all these equations, which then become invisible to the user. While hidden in the worksheet, equations in the collapsed area continue to calculate in sequence. Inside the collapsed area is a Solve Block, which is used to solve for the eigenvalue $k$.

Mathcad uses the supplied guess value to initiate its solution finding process. In the present problem, more than one wave mode is possible owing to the layering of the fluid system. The problem admits multiple solutions, and which one of them is sought depends on the initial value. Therefore, it is important for the user to choose
an appropriate guess value of $k$ in order to get the desired wave mode. We have suggested in the worksheet four optional guesses corresponding to different wave types. The first one is the explicit dispersion relation by Eckart [1951], which gives an approximate wavenumber for linear waves in a single inviscid layer. The second and the third ones are for the limits of deep and shallow waves, respectively. If none of them lead to the desired wave mode, the user may attempt to use an ad hoc guess value of his own choice. The guess value is then converted into complex in order to find a complex solution. More details about the worksheet are available in Chiu [2007].

3. Verification

3.1. One-layer bed

We have extensively tested the worksheet for its performance and, in particular, numerical accuracy. We first compare results with previous studies on the damping of water waves over a single layer of mud. These studies include Dalrymple and Liu [1978] and Ng [2000], for mud modeled as a viscous fluid, and MacPherson [1980], Piedra-Cueva [1993] and Zhang and Ng [2006], for mud modeled as a viscoelastic medium. It is confirmed that our worksheet can produce results that are the same as those presented in these studies. The agreement of results is expected, since the underlying theory is essentially the same in all these studies. For illustration, we select to show, as in Fig. 2, the results generated by us on revisiting two cases previously presented by MacPherson [1980] and Piedra-Cueva [1993]. Figure 2(a) reproduces a portion of Fig. 4 of MacPherson [1980], where $\nu^* \equiv \nu_2/(gd_w^3)^{1/2}$, $D^* \equiv \text{Im}(k)(gd_w)^{1/2}/\sigma$ and $G^* \equiv G_2/\rho_2gd_w$, while Fig. 2(b) is an exact likeness of Fig. 7(b) of Piedra-Cueva [1993], where $X\sigma \equiv d_2(\sigma/\nu_w)^{1/2}$, $Xk_j \equiv \text{Im}(k)d_w$ and $\nu_0 = \nu_2$ in m$^2$/s. MacPherson [1980] considered an inviscid layer over an infinitely deep viscoelastic bed, while Piedra-Cueva [1993] considered a water layer (with a thin water wave boundary layer) over a viscoelastic layer of finite depth. Figure 2(a) displays the effects of bed viscosity and elasticity on the wave attenuation, while Fig. 2(b) shows the wave attenuation as a function of the frequency and the bed viscosity. The input values that we have used to simulate these two cases are given in the figure caption. In MacPherson’s cases, the wave attenuation always decreases as the bed elasticity (or stiffness) increases. For a finite bed layer, the trend is no longer monotonic. As in Piedra-Cueva’s cases, elasticity can lead to the occurrence of resonance at a particular frequency to a finite bed, thereby a dramatic increase in the wave attenuation. Therefore, whether the elasticity is to decrease or to increase the wave damping depends on the wave frequency or the mud depth. It is also worth noting that Fig. 2 shows only the wave mode with the smaller wave attenuation. There is a switch of wave modes at the peak of one of the curves in either case. The dashed extensions represent the wave mode with the higher attenuation. It is near these points of mode switching where the two possible
solutions are very close to each other, and hence care needs to be taken in order to choose a guess value that is close enough to the desired solution. In MacPherson’s case of $G^* = 0$, the guesses $k_{\text{guess}2}$ and $k_{\text{guess}3}$ will lead to the wave mode of lower and higher attenuation, respectively, when $\nu^*$ is small. The reverse is true when $\nu^*$ is large. In Piedra-Cueva’s case of $\nu_b = 0.001$ m$^2$/s, it is the shorter wave mode (i.e. larger Re($k$)) that decays slower when $X\sigma$ is small; the opposite is true when $X\sigma$ is large.

Fig. 2. Results generated by the Mathcad worksheet to reproduce (a) a portion of Fig. 4 of MacPherson [1980]; (b) Fig. 7(b) of Piedra-Cueva [1993]. In (a): $T = 12.6875$ s, $\rho_w = 1.000$ kg/m$^3$, $\rho_2 = 2.000$ kg/m$^3$, $d_w = 10$ m, $d_1 = 0$, $d_2 = 100$ m, and $\nu_w = 10^{-10}$ m$^2$/s. In (b): $d_w = 0.3$ m, $d_1 = 0$, $d_2 = 0.09$ m, $\rho_w = 1.000$ kg/m$^3$, $\rho_2 = 1.370$ kg/m$^3$, $G_2 = 100$ Pa, and $\nu_w = 10^{-6}$ m$^2$/s; $\nu_b = \nu_2$ in m$^2$/s. The dashed extensions represent the wave mode with a larger attenuation rate.
3.2. Two-layer bed

We next compare results with Maa and Mehta [1990], who have performed laboratory flume tests on the interaction between water waves and a partially consolidated bed with depth-varying properties. They had conducted rheometric experiments [Maa and Mehta, 1988] to determine for each test run an empirical relationship between the viscoelastic parameter and the dry density of mud. Based on measured density profiles, they further derived empirical correlations of the dry density with depth below the mud surface. On applying their multi-layer model to the test cases, they discretized the bed into four layers, each with distinct constant density, viscosity and shear modulus. They explained that selection of the number of mud layers and the layer thicknesses was guided by the need for adequately simulating the depth variation of the density.

We here attempt to redo their model simulations, but with the bed discretized into two layers instead. The objective is to find out how this reduction in the bed layering will affect the modeling results when compared with the measured data. We have chosen the following way of forming the two layers in our model in order to sufficiently reflect the depth varying of the bed properties. The upper/lower bed layer is formed by merging the top/bottom two layers in the model of Maa and Mehta [1990]. The properties of the upper layer, and of the lower layer, are given by those on the interface between the first and second layers, and on the interface between the third and fourth layers, respectively, in the model of Maa and Mehta [1990]. We summarize in Table 1 the input data that we have used in our worksheet for the simulation of 13 test runs. These data have been compiled based on the information provided in Maa [1986], and Maa and Mehta [1987, 1990].

Figure 3 comprises two plots comparing the modeling results on the wave attenuation with the data measured by Maa and Mehta [1990], where the upper plot

Table 1. Input data for simulation of the test runs by Maa and Mehta [1990], where $\mu_j = \rho_j \nu_j$ ($j = w, 1, 2$). Other inputs are: $\rho_w = 1,000$ kg/m$^3$, $\mu_w = 0.001$ Pas.

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Fig. 3. Comparison between predicted and measured wave attenuation: (a) prediction by the present model; (b) prediction by the multi-layer model of Maa and Mehta [1990].

shows the prediction by our model, and the lower plot shows the prediction by Maa and Mehta [1990] themselves. The dotted diagonal is inserted to help judge the agreement between prediction and measurement. It is remarkable that, despite simplification in bed layering, our model can generate results that are comparable with those by Maa and Mehta [1990] in terms of agreement with experiment. We further show in Fig. 4 the velocity and pressure amplitude profiles for Run 5–2, as predicted using our model (solid lines), and by Maa and Mehta [1990] (dashed lines), with some data measured by Maa and Mehta [1990] (symbols). It is obvious that our two-layer bed model can predict profiles very close to those predicted by the four-layer bed model of Maa and Mehta [1990]; the differences are practically
insignificant. Again, each model exhibits a comparable degree of agreement between prediction and measurement.

4. Summary

A Mathcad worksheet has been developed as a handy tool to help engineers evaluate properties (wavenumber, wave attenuation, interfacial wave amplitudes, velocity and pressure amplitude profiles) of a progressive surface gravity wave propagating in water over a two-layer muddy bed. In each bed layer, the mud is modeled as a
viscoelastic Voigt medium with constant viscosity and shear modulus of elasticity. Our worksheet has been extensively tested for its numerical accuracy by comparing results with previous studies on waves over either a one-layer or a multi-layer bed. The worksheet is of value not only to practicing engineers, but also to researchers in the areas of coastal engineering, wave mechanics, and so on. The Mathcad worksheet file is available upon request from the first author.

Users are of course cautioned that the results produced by the Mathcad worksheet are only as good as the theory itself, which in practice must be subject to bounds of applicability (e.g. the present theory may not work well when the wave is strongly nonlinear, or when the bed materials exhibit strongly non-Newtonian rheological behaviors, and so on). The worksheet presented here is only a first version of its kind; future versions with extended capabilities are expected. As the knowledge of the problem advances in the future, or as demanded by the industry, any user can readily make changes to the worksheet in order to suit one’s specific needs, thanks to the open and user-friendly Mathcad environment.

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References

Chiu, H. S. [2007] Water Waves in a Multi-Layer Fluid System, FYP Report, Department of Mechanical Engineering, The University of Hong Kong, Hong Kong.


