Electroosmotic Flow Through a Circular Tube With Slip-Stick Striped Wall

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This is an analytical study on electrohydrodynamic flows through a circular tube, of which the wall is micropatterned with a periodic array of longitudinal or transverse slip-stick stripes. One unit of the wall pattern comprises two stripes, one slipping and the other nonslipping, and each with a distinct z potential. Using the methods of eigenfunction expansion and point collocation, the electric potential and velocity fields are determined by solving the linearized Poisson–Boltzmann equation and the Stokes equation subject to the mixed electrohydrodynamic boundary conditions. The effective equations for the fluid and current fluxes are deduced as functions of the slipping area fraction of the wall, the intrinsic hydrodynamic slip length, the Debye parameter, and the z potentials. The theoretical limits for some particular wall patterns, which are available in the literature only for plane channels, are extended in this paper to the case of a circular channel. We confirm that some remarks made earlier for electroosmotic flow over a plane surface are also applicable to the present problem involving patterns on a circular surface. We pay particular attention to the effects of the pattern pitch on the flow in both the longitudinal and transverse configurations. When the wall is uniformly charged, the adverse effect on the electroosmotic flow enhancement due to a small fraction of area being covered by no-slip slots can be amplified if the pitch decreases. Reducing the pitch will also lead to a greater deviation from the Helmholtz–Smoluchowski limit when the slipping regions are uncharged. With oppositely charged slipping regions, local recirculation or a net reversed flow is possible, even when the wall is on the average electropositive or neutral. The flow morphology is found to be subject to the combined influence of the geometry of the tube and the electrohydrodynamic properties of the wall.

Keywords: electroosmotic flow, hydrodynamic slippage, microchannel flow

1 Introduction

The development of microelectromechanical systems (MEMS) has been advancing in a rapid pace owing to the versatile applications and unique merits of these systems; for instance high compactness of devices, reduced sample volumes and scaled-down analysis times. Electrophoresis, the generation of effective liquid motion through the interactions of ions in the electric double layer (EDL) with an externally applied electric field, serves as a key driving mechanism in MEMS. Burgreen and Nakache [1] were among the first to analytically study electroosmotic flow (EOF) in very fine rectangular channels, while Rice and Whitehead [2] studied EOF in a circular capillary. The assumption of uniform z potential was later relaxed by Anderson and Idol [3], who, in the thin EDL limit, developed analytical solutions for flow through a circular channel with axially varying z potential. A nonuniform distribution of wall potential may lead to various effects on the flow and the associated mass transport, as has been investigated analytically or experimentally by various authors [4–8].

Meanwhile, superhydrophobicity is another evolving topic of research in microfluidics given its promising efficacy in drag reduction [9–14]. Among many other causes, surface heterogeneity plays a central role in determining hydrodynamic effective slippage [15–17]. Philip [18,19] investigated flows over porous media by treating the solid-fluid and fluid-fluid interfaces as no-slip and no-shear boundaries respectively. He solved for flows satisfying these mixed boundary conditions, in particular for that in a circular tube with longitudinal slip-stick patterns. Employing a different analytical approach, Lauga and Stone [20] revisited the problem by further considering a tube with periodic transverse distributions of wall slip. Their problem was recently extended by Ng and Wang [21], who presented a model for flow through a periodically grooved tube without subject to the Cassie state assumption.

Following the pioneering work of Muller et al. [22], much interest has been generated to probe the possibility of enhancement of EOF by hydrodynamic slippage [23–26]. It was not until recently that the focus has been on surfaces with nonuniform properties. Based on the low-potential assumption and the thin EDL limit, Squires [27] conducted an analytical study of EOF over a heterogeneously patterned surface, which led him to make remarks particularly for flow over two types of patterns. First, when the wall potential is everywhere constant, the EOF is enhanced by exactly the same amount as if the surface were uniformly slipping with slippage equal to the effective slip length. Second, EOF over a surface with uncharged regions of perfect-slip is the same as if the surface were uniformly charged but nonslipping. More recent studies [28–30] looked into the regime of arbitrary EDL thickness. Belyaev and Vinogradova [31] and Ng and Chu [32] found that EOF will always be inhibited if the uncharged regions are not perfect- but partial-slippering. This flow inhibition exists even for uncharged regions of perfect-slip if the EDL is not sufficiently thin. Furthermore, Belyaev and Vinogradova [31] showed that with oppositely charged regions, flow reversal is possible even on an electroneutral surface under the effect of hydrodynamic slip. Ng and Chu [32] deduced some general results for EOF through a...
plane channel of arbitrary height with striped walls. The problem was recently extended to unsteady EOF by Chu and Ng [33], who considered EOF driven by oscillatory pressure gradient and electric field of the same frequency in a parallel-plate channel.

The present study aims to study electrokinetic flow in a circular microchannel with periodically distributed stripes of different wall potentials and hydrodynamic slip lengths. We shall derive the section-averaged effective equations for flow under mechanical and electrokinetic forcings in a circular channel with alternating no-slip and partial-slip stripes of distinct z potentials aligned either longitudinal or transverse to the flow. Under the condition of low potential, the electric potential of the EDL is found from the linearized Poisson–Boltzmann (P–B) equation. The solution of the potential and the velocity fields are expressible by eigenfunction expansions, where the coefficients are computed numerically by imposing the heterogeneous electric and hydrodynamic conditions at discrete points on the boundary. Theoretical limits concerning several types of wall patterns are obtained analogous to those for electrokinetic flows over an inhomogeneous surface or in a micropatterned channel [27–33]. In this work, we pay attention particularly to the effect of period length or pitch of the pattern on the EOF, where the pattern can be aligned parallel or transverse to the flow. Increasing the number of stripes or shortening the period length will diminish the hydrodynamic effective slip length, which may result in dissimilar effects on the EOF depending on the pattern configurations. First, when the wall is uniformly charged, the appreciable reduction in EOF enhancement caused by the presence of a very small fraction of area being no-slip, as discussed by Ng and Chu [32], can be intensified. Second, when the perfectly-slipping regions are uncharged, a smaller period will lead to an earlier departure from the Helmholtz–Smoluchowski limit for the same slipping area fraction. The problems made by Squires [27] are still applicable here. Inspired by Belyaev and Vinogradova [31], we further look into the interesting scenario involving oppositely charged slipping regions. We demonstrate that, under the combined effect of charge and slippage variations, various convective flow patterns can be generated, including recirculation cells, flow reversal, large slip-induced EOF enhancement, and so on. We remark that the flow morphology depends sensitively on the distribution of properties on the channel wall. The channel confinement also alters the recirculation of fluid as well as the possible flow reversal in a tube.

2 The Problems

Consider Stokes flow of an incompressible Newtonian fluid through a circular tube, of which the wall is patterned with a periodic array of alternating slipping and nonslipping stripes of different wall potentials. The wall is located at \( r = R \), where \( r \) is the radial coordinate, and \( R \) is the radius of the tube. The stripes are aligned either parallel or transverse to the flow along the tube.

For the longitudinal-stripe-alignment shown in Fig. 1(a), there are \( K \) periodic units of stripes on the tube wall. One unit of the

Fig. 1 Electrokinetic flow through a tube of radius \( R \), where the wall is patterned with a periodic array of (a) longitudinal stripes or (b) transverse stripes. One unit of wall pattern consists of a nonslipping stripe of \( \zeta \) potential \( \zeta_{NS} \) and a slipping stripe of slip length \( \lambda \) and \( \zeta \) potential \( \zeta_{S} \). The radial, angular and axial coordinates are respectively \( (r, \theta, z) \) and their corresponding velocity components are denoted by \( (u, v, w) \). With longitudinal stripes, the flow is unidirectional and purely along the \( z \)-direction; and with transverse stripes, the flow is two-dimensional in the \( (r, z) \) plane. In (a), one periodic unit is from \( \theta = 0 \) to \( \theta = \pi \), and \( K \) is the number of periodic units on the wall. In (b), one periodic unit is from \( \theta = 0 \) to \( \theta = 2\pi \). In either case, the area fraction of the slipping region is denoted by \( a \).
wall pattern, whose period arc length is $2\pi R/K$, consists of a slipping stripe of arc length $2\pi R/K$, hydrodynamic slip length $\lambda$, and wall potential $\zeta_w$, and a nonslipping stripe of arc length $2(1 - a)\pi R/K$ and wall potential $\zeta_{NS}$. The number $0 \leq a \leq 1$ denotes the area fraction of the wall which is slipping. By symmetry, it suffices to consider half a period, $0 \leq \theta \leq \pi/K$, in the analysis. For the transverse-stripe-alignment shown in Fig. 1(b), one unit of the wall pattern, which is of a period length $2L$, consists of a slipping stripe of $2aL$, slip length $\lambda$, and wall potential $\zeta_w$ and a nonslipping stripe of $2(1 - a)\pi R/K$ and wall potential $\zeta_{NS}$. Again, the number $0 \leq a \leq 1$ denotes the area fraction of the wall which is slipping. In both cases, the intrinsic or microscopic slip length is allowed to vary in the range $0 \leq \lambda \leq \infty$, where the two limiting values $\lambda = 0$ and $\lambda = \infty$ correspond to non-slip and perfect-slip conditions, respectively. The pattern period length or pitch is assumed to be comparable with the radius of the tube.

3 Longitudinal Stripes

In this configuration, stripes are aligned parallel to the flow as shown in Fig. 1(a). Assuming low potentials and nonoverlapped EDL, the Debye–Hückel (D–H) approximation is used to linearize the P-B equation for the electric potential $\psi(r, \theta)$ to

$$\nabla^2 \psi = \kappa^2 \psi$$  \hspace{1cm} (1)

where $\nabla$ is the inverse of the Debye length (Debye parameter), $\zeta_0$ is the valence of the $z_0$ ion, $\kappa$ is the modified Bessel function of the first kind of order $k = \zeta_0 z_0$, $e$ is the elementary charge, $n_0$ is the bulk concentration of the ions at the neutral state, and $T$ is the absolute temperature. The D–H approximation is valid for $\kappa z_0 < 1$.

For the present problem, $\psi$ is even in $\theta$. The solution to Eq. (1) is expressible by

$$\psi(r, \theta) = A_0 I_0(kr) + \sum_{n=1}^\infty A_n \cos(nkr) J_0(kr)$$

where $A_n$ is the modified Bessel function of the first kind of order $n$. The coefficients $A_n, n \geq 1\ldots$ are to be determined on applying the boundary condition

$$\psi(r, \theta) = A_0 + \sum_{n=1}^\infty A_n \cos(nkr) = \begin{cases} \zeta_w & 0 < \theta < \pi < K \\ \zeta_{NS} & \pi < \theta < \pi/K \end{cases}$$  \hspace{1cm} (3)

where the Fourier series coefficients are found to be

$$A_0 = a\zeta_w + (1 - a)\zeta_{NS}, \quad A_n = 2(\zeta_w - \zeta_{NS}) \frac{\sin(n\pi \gamma)}{n\pi}$$

The charge density $\rho_v(r, \theta)$ can then be found by

$$\rho_v(r, \theta) = -e \nabla \psi = -ek^2 \psi$$  \hspace{1cm} (5)

The electric potential $\psi(r, \theta)$ is the potential due to the EDL at the equilibrium state. In the presence of an externally applied electric field $E_{ext} = (0, 0, E_z)$ in the Cartesian coordinates, where $E_z$ is a constant, the total potential is

$$\psi_{tot}(r, \theta, z) = \psi(r, \theta) - (r \cos \theta, r \sin \theta, z) \cdot E_{ext}$$

Owing to the free charges in the fluid, a Lorentz body force term $\rho_v \vec{E}$ is added to the momentum equation, where $\rho_v$ is the charge density given by Eq. (5), and

$$\vec{E}(r, \theta, z) = -\left( \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial \theta} \right) \psi_{int} = -\left( \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial \theta} \right) \psi + \vec{E}_{ext}$$  \hspace{1cm} (7)

is the total electric field strength. The present work is based on the low-Dukhin limit, by which we may assume that the applied electric field and the resulting flow field will not disturb the EDL significantly from its equilibrium state. We also consider the Reynolds number being so low that Stokes flow can be assumed. The governing Stokes equation is solved by the method of eigenfunction expansion, where the coefficients are determined by point-matching the mixed-type boundary conditions.

Before we proceed, let us introduce the following normalization:

$$\begin{cases} (\hat{r}, \hat{\lambda}) = (r, \lambda)/R \\ \hat{\kappa} = \kappa R \\ (\hat{\psi}, \zeta_w, \hat{A}_0, \hat{A}_n) = (\psi, \zeta_w, A_0, A_n)/\zeta_{NS} \end{cases}$$

by which

$$\hat{A}_0 = a\zeta_w + (1 - a), \quad \hat{A}_n = 2(\zeta_w - \zeta_{NS}) \frac{\sin(n\pi \gamma)}{n\pi}$$

which will be used in our later derivations.

Under a constant pressure gradient $P_c = dp/dz$ and a constant electric field $E_{ext} = (0, 0, E_z)$ both applied purely in the axial direction, the flow which fully-developed is unidirectional in this direction, and the axial velocity $w(r, \theta)$ is governed by

$$\mu \nabla^2 w - P_c + \rho_v E_z = 0$$

where $\mu$ is the dynamic viscosity of the fluid. Linearity of the equation allows us to decompose the velocity into components due solely to either the pressure or the electric forcing: $w = w_{po} + w_{EO}$. By convention, the two types of flow are termed Poiseuille (PO) flow and electroosmotic (EO) flow, respectively. The two forcings give rise to the following velocity components, which are even in $\theta$:

$$w_{po}(r, \theta) = \left[ \frac{1}{4} (1 - r^2) + \sum_{n=1}^\infty B_n \cos(nkr) \right] P_c^*$$

where

$$P_c^* \equiv -\frac{R^2}{\mu} P_c \quad \text{and} \quad E_z^* \equiv -\frac{\zeta_{NS}}{\mu} E_z$$

are the forcing parameters with dimensions of velocity.

The parameter $\delta_1$ in Eq. (12) is the dimensionless (normalized by $K$) hydrodynamic effective or macroscopic slip length for flow parallel to the stripes. The coefficients $\delta_1, B_n, \hat{C}_0, \text{and } \hat{C}_8$ are to be determined on applying the mixed-type boundary conditions

$$w = \begin{cases} -\hat{\delta}_w \partial w/\partial \theta & 0 < \theta < \pi < K \\ 0 & \pi < \theta < \pi/K \end{cases}$$

where $\hat{\delta}_w$ is the dimensionless microscopic slip length of the slipping stripes. On substituting the velocities in Eqs. (12) and (13) into the conditions above, the following equations are obtained:

$$\sum_{n=1}^M \hat{B}_n \left[ (1 + nK \hat{\lambda} \cos(nkr) - \cos(n\pi)) \right] = \hat{\delta}_w (0 < \theta < \pi < K)$$

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\[ \text{NOMENCLATURE}\]

\[ \text{Symbols}\]

\[ \text{Greek letters}\]

\[ \text{Subscripts}\]

\[ \text{Superscripts}\]

\[ \text{References}\]

\[ \text{Acknowledgments}\]

\[ \text{Appendices}\]

\[ \text{Results and Discussion}\]

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\[
\sum_{n=1}^{M} \hat{B}_n \cos(nK\theta) - \cos(n\pi) = 0 \quad (\alpha\pi/K < \theta < \pi/K) \quad (17)
\]

\[
\hat{\delta}_1 = -2 \sum_{n=1}^{M} \hat{B}_n \cos(n\pi) \quad (18)
\]

\[
\sum_{n=1}^{M} \hat{C}_n \left[ (1 + nK^2\lambda) \cos(nK\theta) - \cos(n\pi) \right] = -\hat{\kappa}_1 \hat{A}_0 \frac{I_1(\hat{k})}{I_0(\hat{k})} - nK^2 \left( \frac{\hat{I}_{sk\hat{k}}}{\hat{I}_{s\hat{k}}} - nK^2 \right) \cos(nK\theta) - \cos(n\pi) \} \quad (0 < \theta < \alpha\pi/K) \quad (19)
\]

\[
\sum_{n=1}^{M} \hat{C}_n \cos(nK\theta) - \cos(n\pi) = 0 \quad (\alpha\pi/K < \theta < \pi/K) \quad (20)
\]

\[
\hat{C}_0 = -1 - \sum_{n=1}^{M} \hat{C}_n \cos(n\pi) \quad (21)
\]

where we have truncated the series each to \(M\) terms. In the sense of point collocation, we impose the two pairs of Eqs. (16)-(17) and Eqs. (19)-(20) at \(M\) discrete points evenly distributed in the domain \(0 < \theta < \pi/K\) while avoiding the junction point \(\theta = \alpha\pi/K\). Two \(M \times M\) systems of linear equations for the unknowns \(\hat{B}_{1,M}\) and \(\hat{C}_{1,M}\) are formed and can be solved numerically by a standard routine. The coefficients \(\hat{\delta}_1\) and \(\hat{C}_0\) are then found from Eqs. (18) and (21).

Taking average over a periodic unit of cross-sectional area, we get the mean velocity for the flow

\[
\bar{w} = L_{11}^{\parallel} P^*_z + L_{12}^{\parallel} E^*_z \quad (22)
\]

are dimensionless phenomenological coefficients known as the hydrodynamic conductance and the streaming flow conductance, respectively. The driving forces \(P^*_z\) and \(E^*_z\) are defined in Eq. (14), where \(E^*_z\) is also known as the Helmholtz–Smoluchowski velocity. Note that we have expressed the results in such a way that the forcings have the same dimensions as the response, while the phenomenological coefficients are all nondimensional. These expressions provide analytical convenience for identifying the significance of each type of wall pattern. In the above expression for the hydrodynamic conductance, the first term corresponds to the basic Poiseuille flow and the second term corresponds to the flow enhancement by slip. In the expression for the streaming conductance, the basic EOF is modified by nonuniform wall potential and slippage through the coefficients \(\hat{A}_0\) and \(\hat{C}_0\) respectively. One should note that the dimensionless phenomenological coefficients are the response per unit value of the corresponding forcing. Hence, the effect of surface pattern can be studied simply through these coefficients. In particular, we shall use the streaming conductance to describe EOF in the discussion of our results.

The axial electric current density arising from conduction and convection is given by

\[
I_z = \sigma E_z + \rho_e w \quad (25)
\]

where \(\sigma\) is the electric conductivity of the fluid. The electric conductivity can reasonably be assumed to be a constant under the condition of low potentials [34]. On substituting Eqs. (5), (12), and (13) for \(\rho_e\) and \(\omega\), and taking average over a unit sectional area, we get the following expression for the mean current density:

\[
L_z = L_{12}^{\parallel} P^*_z + L_{22}^{\parallel} E^*_z \quad (26)
\]

where

\[
P^*_z = -\frac{\varepsilon_{NS}^2 \varepsilon_k}{\mu} P^*_z \quad \text{and} \quad E^*_z = \frac{\varepsilon_k^2 \varepsilon_0}{\mu} E^*_z \quad (27)
\]

are the forcing parameters with dimensions of current density. Note that they are so defined that their ratio is equal to that of with the dimension of velocity: \(E_z^*/P_z^* = E_z^*/P_z^*\). The dimensionless phenomenological coefficients \(L_{12}^{\parallel}\) and \(L_{22}^{\parallel}\) are respectively known as the streaming current conductance and the electrical conductance, which can be determined by numerical integration. Equations (22) and (26) are the advantage relations for the flow and ionic fluxes with longitudinal-stripe-alignment. We note that, in a manner similar to that in Ng and Chu [32], the Onsager reciprocity relation, or the equality of the two nonconjugate coefficients \(L_{12}^{\parallel} = L_{21}^{\parallel}\), can be proved. To avoid repetition, the proof is omitted here.

3.1 Uniform Wall Potential. Let us look into three particular cases. First, consider the case when the wall potential is uniform so that \(\zeta_{NS} = \zeta_S = 0\) or \(\zeta_S = 1\). It follows that \(A_0 = 1\) and \(\hat{A}_0 = 0\). From Eqs. (16), (17), (19), and (20), \(\hat{B}_n\) and \(\hat{C}_n\) are found to be interrelated by \(\hat{C}_n = -2\hat{k}\frac{I_1(\hat{k})}{I_0(\hat{k})}\hat{B}_n\). Furthermore, using Eqs. (18) and (21), we get \(\hat{C}_0 = -\hat{\kappa}_1 \hat{A}_0 I_1(\hat{k})/I_0(\hat{k}) - 1\). Substituting this into Eq. (24), we have the streaming flow conductance be given by

\[
L_{12}^{\parallel} = 1 + \hat{\kappa}_1 \frac{I_1(\hat{k})}{I_0(\hat{k})} - \frac{2}{\hat{\kappa}_1} \frac{I_1(\hat{k})}{I_0(\hat{k})} \quad \text{for} \quad \hat{\zeta}_S = 1 \quad (28)
\]

In terms of dimensional quantities, the mean EOF velocity is given by

\[
\bar{w}_{EOF} = \left[ 1 + \hat{\kappa}_1 \frac{I_1(kR)}{I_0(kR)} - \frac{2 k}{\hat{\kappa}_1} \frac{I_1(kR)}{I_0(kR)} \right] \left( -\frac{\varepsilon_k}{\mu} E_z \right) \quad \text{for} \quad \zeta_{NS} = \zeta_S = \zeta \quad (29)
\]

Note that the EOF is enhanced by the slip through the term \(\kappa_0 [I_1(kR)/I_0(kR)]\), which is linearly proportional to the effective slip length \(\hat{\delta}_1\). In the limit of a very thin EDL, \(k \gg 1\), Eq. (28) gives \(L_{12}^{\parallel} \approx 1 + k \hat{\delta}_1\). This result has been derived previously by Squires [27], who remarked that EOF with a constant wall potential over an arbitrarily slipping surface is enhanced by precisely the same amount as would be found by assuming the effective slip length to apply homogeneously. Our result becomes identical to his when the EDL is very thin compared with the radius of the tube. Ng and Chu [32] studied the problem of electrokinetic flows through a parallel-plate channel with stripes of partial-slip and derived a more general result which holds for arbitrary channel height and EDL thickness as long as the EDLs are not strongly overlapped,

\[
L_{12}^{\parallel} = 1 + \kappa_0 \frac{\tanh(kh)}{kh} \quad \text{for} \quad \zeta_S = \zeta_{NS} = \zeta \quad (30)
\]

where \(h\) is half the channel-height. We have here obtained an analogous expression for a circular tube, and Squires’ remark still holds. One can find that Eq. (28) works as well for homogeneous walls with uniform boundary slip [23,24] with the replacement of the effective slip length by the uniform slip length of the homogeneous walls. The effective slip length that has been found in the hydrodynamic problem can be used directly in the electrokinetic problem as if the slip were homogeneous, provided that the wall potential is constant.

3.2 Very Thin EDL. Second, let us consider, for any values of the wall potentials, the limiting condition of a very thin EDL


\[ \kappa \gg 1. \] From Eq. (10), we get \[ \sum_{n=1}^{\infty} \hat{A}_n \cos(nK\theta) = \hat{\zeta}_S - \hat{A}_0 \]
where \( 0 < \theta < a\pi/K, \) and \( \sum_{n=1}^{\infty} \hat{A}_n \cos(n\pi) = 1 - \hat{A}_0. \) Under this
limiting condition, the right-hand side of Eq. (19) is simplified to \( 1 - \hat{\zeta}_S(1 + \kappa\hat{\lambda}) \), which is a constant independent of \( \theta. \) Then from
Eqs. (16), (17), (19), and (20), \( \hat{B}_1, \) and \( \hat{C}_0 \) are nearly collinear
vectors, related by \( \hat{C}_0 \approx \hat{B}_1 \approx \hat{D}_1 \approx \hat{D}_0 = 0. \) It follows from
Eqs. (18), (21), and (24) that

\[ L_{||} = -\hat{C}_0 \approx 1 + \frac{\hat{B}_0}{2} \left[ \hat{\zeta}_S(1 + \kappa\hat{\lambda}) - 1 \right] \quad \text{for} \quad \kappa \gg 1 \] (31)

which agrees with the expression obtained by Ng and Chu [32], who considered EOF in a channel of finite height with stripes of partial-slip. In fact, Eq. (31) is independent of the dimension of the channel. This implies that, as far as the limit of very thin EDL is concerned, the theoretical limits derived hold for channel of any geometry.

Some particular cases readily follow from Eq. (31). For equal wall potentials \( \hat{\zeta}_S = 1, \) we get \( L_{||} \approx 1 + \kappa\hat{\lambda}, \) which recovers the result in Sec. 3.1. For perfect-slip stripes \( \hat{\lambda} = \infty, \) we get \( L_{||} \approx 1 + \kappa\hat{\lambda}; \) as has been previously deduced by various authors [27–32] who considered EOF over an inhomogeneous surface and in a micropatterned channel. If the perfect-slip region is uncharged, \( \hat{\zeta}_S = 0, \) the result further reduces to \( L_{||} \approx 1. \)

In other words, in this limiting case, the EOF velocity is simply equal to the Helmholtz–Smoluchowski velocity without seeing any slip effect on it

\[ \hat{w}_{\text{EO}} = -\frac{\hat{\zeta}_S}{\mu} E_z \quad \text{for} \quad \hat{\zeta}_S = 0, \quad \hat{\lambda} = \infty, \quad \kappa \gg 1 \] (32)

This interesting result was first pointed out by Squires [27], who remarked that surfaces with uncharged perfect-slip regions will give rise to no enhancement by slip, instead giving precisely the same EOF as if the surface were nonslipping and homogeneously charged.

### 3.3 Uniformly No-Slip Walls With Heterogeneous Wall Potential

Anderson and Idol [3] were the first to theoretically study EOF through a circular pore with axially (transversely) varying wall potential in the thin EDL limit. Although in this section wall potential is varying with radial position, the result here surprisingly agrees with theirs. Here, we consider the limiting case where the walls are uniformly nonslippering but with discrete regions of differing wall potentials. From Eq. (10), we get

\[ \sum_{n=1}^{\infty} \hat{A}_n \cos(nK\theta) - \cos(n\pi) = \hat{\zeta}_S - 1 \quad \text{for} \quad 0 < \theta < a\pi/K, \]

\[ \sum_{n=1}^{\infty} \hat{A}_n \cos(n\pi) = 0 \quad \text{for} \quad a\pi/K < \theta < \pi/K. \]

In conjunction with Eqs. (19)–(21), we obtain the relation \( \hat{C}_0 = -\hat{A}_0. \) Substituting this into Eq. (24), the streaming flow conductance and section-mean EOF velocity are

\[ L_{||} = \hat{A}_0 \left[ 1 - \frac{2I_1(kR)}{k\hat{\lambda}I_0(kR)} \right] \quad \text{for} \quad \hat{\lambda} = 0 \] (33)

\[ \hat{w}_{\text{EO}} = \hat{A}_0 \left[ 1 - \frac{2I_1(kR)}{k\hat{\lambda}I_0(kR)} \right] \left( \frac{\hat{\zeta}_S}{\mu} E_z \right) \quad \text{for} \quad \hat{\lambda} = 0 \] (34)

where \( \hat{A}_0 \) and \( \hat{\lambda}_0 \) are the mean wall potential over a periodic unit given in Eqs. (4) and (9), respectively. In the limit of a very thin EDL, the mean EOF velocity reduces to \( \hat{w}_{\text{EO}} = \hat{A}_0 (\hat{\zeta}_S - \hat{\lambda}_0) \). This echoes the finding of Anderson and Idol [3], which was later confirmed by Long et al. [4] and Ghosal [8] with lubrication theory. They found that the mean EOF velocity within a capillary is given exactly by solving the Helmholtz equation (i.e., Eq. (1)) with the local wall potential replaced by the global average wall potential of the capillary. In other words, the average EOF velocity within a heterogeneous capillary is the same as that of a homogeneous capillary, or the Helmholtz–Smoluchowski velocity, with the uniform wall potential replaced by the mean potential of the nonuniform charged wall. This remark still holds here although the nonuniform wall potential is distributed in an entirely different fashion.

In the next section, we are going to show that our model is able to reproduce results that are consistent with these authors when surface heterogeneity becomes a function of axial position. We shall show that the expressions governing the longitudinal and transverse streaming conductance are the same.

### 4 Transverse Stripes

We next consider the configuration where stripes are aligned normal to the flow as shown in Fig. (1b). The flow in this case is axisymmetric and independent of \( \theta. \) Based on the same assumptions made in the preceding section, the electric potential \( \psi(r, z) \) of the EDL is governed by the linearized P–B equation. The solution is expressible by

\[ \psi(r, z) = D_0 \left( I_0(\kappa r) + \sum_{n=1}^{\infty} D_n \cos(n\pi z) \right) \frac{I_0(\kappa_0 r)}{I_0(\kappa_0 R)} \] (35)

where \( \kappa_0 = n\pi/L, \) and \( \kappa_0^2 = x^2 + y^2. \) The mixed boundary conditions are

\[ \psi(r, z) = D_0 + \sum_{n=1}^{\infty} D_n \cos(n\pi z) \left( \hat{\zeta}_S \frac{\sin(n\pi z)}{n\pi} = 0 < z < aL \right. \]

\[ \left. aL < z < L \right. \] (36)

from which the coefficients \( D_0, D_n \) are found as follows:

\[ D_0 = n\pi \hat{\zeta}_S(1 - a) \hat{\zeta}_S, \quad D_n = 2(\hat{\zeta}_S - \hat{\zeta}_S) \frac{\sin(n\pi z)}{n\pi} \] (37)

Introducing normalization as in Eq. (8), Eqs. (36) and (37) become

\[ \hat{D}_0 + \sum_{n=1}^{\infty} \hat{D}_n \cos(n\pi z) = \left( \frac{\hat{\zeta}_S}{\hat{\zeta}_S} - 1 \right) \frac{\sin(n\pi z)}{n\pi} \] (38)

where \( (\hat{z}, \hat{L}) = (z, L)/R, \)

\[ \hat{D}_0 = \left( D_0/D_n \right) \hat{\zeta}_S, \quad \hat{z}_n = n\pi/L, \]

\[ \hat{\beta}_n^2 = x^2 + y^2 \]

and

\[ \hat{D}_0 = a\hat{\zeta}_S + 1 - a, \quad \hat{D}_n = 2(\hat{\zeta}_S - 1) \frac{\sin(n\pi z)}{n\pi} \] (39)

In fact, from Eqs. (4) and (37), one finds that the coefficients for the potential solution are the same for the longitudinal and transverse configurations.

Again, consider flow driven by a constant pressure gradient and a constant electric field both purely in the axial direction. The present configuration of transverse stripes necessitates the flow to be two-dimensional in the \( (r, z) \) plane under mass conservation. The velocity \( u(r, z) \equiv (u(r, z), w(r, z)) \) is governed by

\[ \nabla \cdot \hat{u} = 0 \]

\[ \nabla^2 \hat{u} - \nabla p = -\left( (\hat{w}_{\text{EO}} + \hat{u}_{\text{EO}}) \right) \]

where \( \nabla \equiv (\partial/\partial r, \partial/\partial z), \)

\[ \nabla \psi = (0, \hat{E}_z), \]

and the total electric field strength is \( \hat{E}_z = -\nabla \psi = (0, \hat{E}_z) \) as stated in Eq. (7) in which \( \psi \) is given in Eq. (35). The primitive variables \( (\mu, w, p) \) are periodic functions of \( z \) as a result of the periodic slip-boundary conditions. By symmetry of the flow field about \( z = 0, \) the radial velocity \( u \) is odd in \( z, \) the axial velocity \( w \) is even in \( z, \) and the pressure \( p \) is odd in \( z. \)

Let us decompose the flow into the Poiseuille (PO) and the electroosmotic (EO) components: \( \hat{u} = \hat{u}_{\text{EO}} + \hat{u}_{\text{PO}}. \) The corresponding momentum equations are
\[ \mu \nabla^2 u_{PO} - \nabla p_{PO} - (0, P_z) = 0 \] (42)

and

\[ \mu \nabla^2 u_{EO} - \nabla p_{EO} + \rho_e \tilde{E} = 0 \] (43)

The hydrodynamic velocity components \( \hat{u}_{PO} = (u_{PO}, w_{PO}) \) are readily found to be

\[
\hat{u}_{PO}(\hat{r}, \hat{z}) = \left\{ \sum_{n=1}^{M} \hat{F}_n \sin(\hat{z}_n \hat{z}) \left[ I_1(\hat{z}_n \hat{r}) - \hat{r} I_0(\hat{z}_n \hat{r}) \right] \right\} \hat{P}_z^n
\]

\[ w_{PO}(\hat{r}, \hat{z}) = \left\{ \frac{1}{4} (1 - \hat{r}^2) + \frac{\hat{\delta}_l}{2} + \sum_{n=1}^{\infty} \hat{F}_n \cos(\hat{z}_n \hat{z}) \left[ 1 - 2 I_1(\hat{z}_n \hat{r}) \right] \right\} \hat{P}_z^n \] (44)

where \( \hat{\delta}_l \) is the dimensionless effective hydrodynamic slip length for transverse stripes on the wall and \( \hat{P}_z^n \equiv -(R^2 / \mu) \hat{P}_z \) is the pressure forcing parameter with dimensions of velocity.

\[ w_{EO}(\hat{r}, \hat{z}) = - \left\{ D_0 \frac{I_0(\hat{r} \hat{k})}{\hat{I}_0(\hat{k})} + G_0 \right\}
+ \sum_{n=1}^{M} \hat{D}_n \cos(\hat{z}_n \hat{z}) \left[ 1 - 2 I_1(\hat{z}_n \hat{r}) \right] \right\} \hat{P}_z^n \] (48)

where \( D_0, \hat{D}_n \) are given in Eq. (39) and \( \hat{E}_z^n \equiv (-v_{NS} / \mu) \hat{E}_z \) is the electric forcing parameter with dimensions of velocity.

Again, we have expressed the forces in such a way that the coefficients have the same dimensions as the response, while the coefficients are all non-dimensional. In the solutions above, the coefficients \( \hat{\delta}_l, \hat{F}_n, G_0, \) and \( \hat{G}_n \) are to be determined on applying the slip-stick boundary conditions

\[ w = \begin{cases} -\hat{\delta}_l \hat{W} / \partial \hat{z} & 0 < \hat{z} < aL \\ 0 & aL < \hat{z} < \hat{L} \end{cases} \] (49)

On substituting the velocities in Eqs. (45) and (48) into the conditions above, the following equations are obtained:

\[ \sum_{n=1}^{M} \hat{G}_n \left[ 1 - 2 I_1(\hat{z}_n \hat{r}) \right] \hat{P}_z^n \left[ \cos(\hat{z}_n \hat{z}) - \cos(n \pi) \right] = -D_0 \hat{k} \hat{k} I_1(\hat{k}) \hat{I}_0(\hat{k}) \] (50)

\[ -2 \hat{\delta}_l \hat{F}_n \frac{I_1(\hat{z}_n \hat{r})}{\hat{I}_0(\hat{z}_n)} \cos(\hat{z}_n \hat{z}) = \frac{\hat{\delta}_l}{2} \] (51)

\[ \hat{\delta}_l = -2 \sum_{n=1}^{M} \hat{F}_n \left[ 1 - 2 I_1(\hat{z}_n \hat{r}) \right] \hat{P}_z^n \left[ \cos(\hat{z}_n \hat{z}) - \cos(n \pi) \right] \] (52)

\[ \sum_{n=1}^{M} \hat{G}_n \left[ 1 - 2 I_1(\hat{z}_n \hat{r}) \right] \hat{I}_1(\hat{z}_n) \hat{I}_0(\hat{z}_n) \cos(\hat{z}_n \hat{z}) = -D_0 \hat{k} \hat{k} I_1(\hat{k}) \hat{I}_0(\hat{k}) \] (53)


\[\sum_{n=1}^{M} \hat{G}_n \left[ \begin{array}{c} 1 - \frac{2 I_1(z_n)}{\beta_n I_0(z_n)} \frac{I_1(z_n)}{I_0(z_n)} \cos(\hat{\theta}_n z) - \cos(n\pi) \end{array} \right] \]

where we have truncated the series each to \( M \) terms. By the method of point collocation, we impose the two pairs of Eqs. (50)–(51) and Eqs. (53)–(54) at \( M \) discrete points evenly distributed in the domain \( 0 < \hat{z} < L \) while avoiding the junction point \( \hat{z} = aL \). Two \( M \times M \) systems of linear equations for the unknowns \( \hat{F}_{1,M} \) and \( G_{1,M} \) are formed and can be solved numerically by a standard routine. The coefficients \( \hat{\delta}_{1} \) and \( \hat{G}_0 \) are then found from Eqs. (52) and (55).

On averaging over the cross-sectional area, we get the mean velocity for the flow

\[\tilde{w} = L_{11} P_{z}^* + L_{12} P_{z}^* \]

where

\[L_{11} = \frac{1}{8} \left(1 + 4 \hat{\delta}_{1}\right)\]

\[L_{12} = -\frac{2 \hat{D}_{0} I_{1}(\hat{k})}{\hat{k} I_{0}(\hat{k})} + \hat{G}_0\]

are, respectively, the hydrodynamic conductance and the streaming flow conductance, both dimensionless.

On substituting Eqs. (5), (45), and (48) for \( \rho_2 \) and \( w \) in the axial electric current density \( I_1 = \sigma E_z + \rho_2 w \), and further taking average over one periodic unit volume that spans in both streamwise and radial directions of the tube, we get the following expression for the mean axial current density:

\[\bar{I}_1 = L_{11} P_{z}^* + L_{12} P_{z}^* + \tilde{w}_{EO} E_z^*\]

where \( P_{z}^* = -(\varepsilon_{NS}/\mu) P_z \), \( E_z^* = -(\varepsilon_{NS}/\mu \hat{D}_0) E_z \) are the hydrodynamic and electrical forcing parameters with the dimension of current density. The streaming current conductance \( L_{12} \) and electrical conductance \( L_{11} \) can be determined by numerical integration. Equations (56) and (59) are the Onsager relations for the flow and ionic fluxes with transverse-stripe-alignment. It can be analytically shown that reciprocity of these relations is satisfied: \( L_{12} = L_{12} \). The proof can be performed in a manner similar to that given in Ng and Chu [32] for flow with longitudinal stripes. The proof is omitted here for the sake of brevity.

Also, for the particular cases that we have looked into for the longitudinal stripes configuration, exactly analogous results can be deduced for the present case of transverse stripes. Without repeating essentially the same details, let us state the results as follows. First, consider the special case when the wall potential is uniform. The streaming flow conductance is found to be given by

\[L_{12} = 1 + \hat{\kappa}_\lambda \frac{I_{1}(\hat{k})}{I_{0}(\hat{k})} \left(1 + \frac{2 I_1(\hat{k})}{\hat{k} I_0(\hat{k})} \right) \]

In terms of dimensional quantities, the mean EOF velocity is given by

\[\tilde{w}_{EO} = \left[1 + \kappa \delta_{\lambda} I_{1}(\kappa \hat{R}) \right] \frac{2 I_1(\kappa \hat{R})}{\kappa \hat{R} I_0(\kappa \hat{R})} \left(1 + \frac{2 I_1(\hat{k})}{\hat{k} I_0(\hat{k})} \right) \]

The EOF is enhanced by the slip through the term \( \kappa \delta_{\lambda} I_{1}(\kappa \hat{R}) \), which is linearly proportional to the effective slip length \( \delta_{\lambda} \). These expressions are analogous to those derived by Ng and Chu [32]. Further, at the limit of a very thin EDL, the EOF velocity in Eq. (61) reduces to

\[\tilde{w}_{EO} \approx \left(1 + \kappa \delta_{\lambda} \right) \left(1 + \frac{2 \varepsilon_{NS}}{\mu} E_z \right) \]

which matches the result by Squires [27]. Hence, in this limit, the EOF is enhanced by a factor equal to the effective slip length divided by the Debye length. Our remarks made earlier for the longitudinal stripes apply to the flow with transverse stripes as well. In particular, the first remark by Squires [27] still holds in the present problem. The effective slip length that has been found in the hydrodynamic problem can be used directly in the EOF formula as if the wall were homogeneous.

Second, consider the limiting case of a very thin EDL, \( \hat{\kappa} \gg 1 \), such that \( \hat{\beta}_0 \approx \hat{\kappa} \). Under this limiting condition, the streaming conductance in Eq. (58) approximates to

\[L_{12} \approx 1 + \frac{\hat{\delta}_{\lambda}}{\hat{\lambda}} \left(1 + \hat{\kappa} \hat{\lambda} - 1\right) \]

If the slip surface is perfectly slipping but uncharged, \( \hat{\lambda} = \infty \) and \( \hat{\delta}_{\lambda} = 0 \), the result further reduces to \( L_{12} \approx 1 \). In other words, in this limiting case, the EOF velocity is simply equal to the Helmholtz–Smoluchowski velocity without subject to any slip effect

\[\tilde{w}_{EO} = -\frac{\varepsilon_{NS}}{\mu} E_z \]

Again, like the flow with longitudinal stripes, the EO plug flow here is not affected by the uncharged perfect-slippping regions.

The next limiting case that we consider is uniform no-slip wall with heterogeneous wall potentials. The streaming flow conductance and section-mean EOF velocity are given by

\[L_{12} = \hat{D}_0 \left[1 - \frac{2 I_1(\hat{k})}{\hat{k} I_0(\hat{k})} \right] \]

\[\tilde{w}_{EO} = \hat{D}_0 \left[1 - \frac{2 I_1(\hat{R})}{\hat{R} I_0(\hat{R})} \right] \left(1 + \frac{2 \varepsilon_{NS}}{\mu} E_z \right) \]

where \( \hat{D}_0 \) and \( \hat{D}_0 \) are the mean wall potential over a periodic unit from Eqs. (37) and (39). In the limit of a very thin EDL, the mean EOF velocity is \( \tilde{w}_{EO} = \hat{D}_0 (-\varepsilon_{NS}/\mu) \). Our results here have two noteworthy implications. First, they agree with the findings by Anderson and Idol [3] and other authors (Long et al. [4], Ghosal [8]); the EOF velocity within a heterogeneous circular tube is simply the Helmholtz–Smoluchowski velocity with the wall potential replaced by the average wall potential over the length of the tube. Second, as mentioned earlier, the coefficients \( \hat{D}_0 \) and \( \hat{D}_0 \) are equal to \( \lambda_0 \) and \( \lambda_0 \) in the problem of longitudinal stripes. One can readily infer from Eqs. (33) and (65) that the expressions for the longitudinal and transverse streaming conductance are the same. It means that, without wall slippage, the average flow is not affected by the spatial distribution of charges on the wall. This has also been shown to be true by Chu and Ng [33] for oscillatory as well as steady EOF through a parallel-plate channel.

5 Results and Discussions

We have used the package MATLAB to solve the \( M \times M \) systems of equations deduced in the problems described above. With
Fig. 2. The longitudinal and transverse streaming conductance, $L_{12}^\parallel$ and $L_{12}^\perp$, as functions of the Debye parameter $\hat{\kappa}$ and the slipping area fraction $a$ where the intrinsic slip length $j = 1$ and wall potential $\zeta_0 = 1$. The number of periodic units for longitudinal stripes $K = 5$ in (a) and (c), while the period length for transverse stripes $L = 1$ in (b) and (d). Dashed lines in (c) and (d) represent the case where $K = 10$ and $L = 5$ respectively. The two insets show the effective slip lengths $\delta_\parallel$ and $\delta_\perp$ as functions of $a$.

numerical results, let us further examine the cases that we have considered in the preceding sections. The input parameters for the computations are the following: the slipping area fraction of the wall $a$, the Debye parameter $\hat{\kappa}$, the intrinsic or microscopic slip length $\lambda$, the $\zeta$ potential of the slipping region $\zeta_0$, and the number of units on the wall $K$ (for longitudinal-stripe-alignment) or the period length of pattern $L$ (for transverse-stripe-alignment).

First, consider the case when the wall is inhomogeneously slipping, but the potential is uniform so that $\zeta_0 = 1$. In this case, the streaming conductance is given by Eq. (28) for the longitudinal configuration of stripes, and Eq. (60) for the transverse configuration. Figures 2(a) and 2(b) show the streaming conductance, $L_{12}^\parallel$ and $L_{12}^\perp$, as functions of $\hat{\kappa}$ and $a$, where $\lambda = 1$ and $\zeta_0 = 1$. For a wall with slippage, the streaming conductance increases almost linearly without bound with $\hat{\kappa}$ for $\hat{\kappa} > 10$. This is in sharp contrast to the case of a no-slip wall, i.e., $a = 0$, for which the streaming conductance at large $\hat{\kappa}$ is upper-bounded by the limit of unity. This observation is consistent with that reported by Ng and Chu [32], who studied a similar problem for EOF in a parallel-plate channel. It is expected since slip-induced EOF in a tube is still enhanced by a factor of $\kappa$ for $\hat{\kappa} \gg 1$, which can indeed be very large when the EDL is much thinner than the effective slip length. When the streaming conductance is plotted against $a$ in Figs. 2(c) and 2(d), our results also exhibit behaviors similar to those reported by Ng and Chu [32]. It is remarkable that slip-induced EOF enhancement can be dramatically diminished by as small as 1% of an otherwise uniformly slipping wall being contaminated by interspersed nonslipping slots. For longitudinal stripes with $K = 5$ (solid lines), a small decrease of $a$ from unity to 0.99 will lead to a drop of 26% and 28% in $L_{12}^\parallel$ for $k = 10$ and $\hat{\kappa} = 5$, respectively. It can be seen that as $K$ increases (dashed lines), such a diminishing effect can be intensified. Explanation can be sought by observing the inset where $\delta_\parallel$ is expressed as a function of $a$. Since an increase in the periodic units of stripe lowers the effective slip length in the range $0 < a < 1$, it essentially steepens the slope as $a$ approaches unity where $\delta_\parallel = \lambda$. Streaming conductance is therefore generally lower for a larger $K$ but it will become independent of $K$ as $a = 1$. On the other hand, for transverse stripes, shortening the period will decrease the effective slip length and increase the slope of the curve near $a = 1$. This observation is consistent with that in the longitudinal case. A larger $K$ in fact amounts to a smaller period arc length. This explains why imposing a larger $K$ in the longitudinal configuration will bring forth the same qualitative effect as shortening the period length in the transverse case.

Second, we examine the case when alternate stripes are perfectly-slipping but uncharged: $\lambda = \infty$ and $\zeta_0 = 0$, and when
the EDL is very thin: \( \kappa \gg 1 \). A formal proof of Squires [27] has led him to remark that this kind of surface will not give rise to any EOF enhancement due to the slip, instead giving precisely the same EOF as if the surface were completely nonslipping and homogeneously charged. Ng and Chu [32] made a conclusion in their parallel-plate channel problem that the thin EDL limit is practically achieved only when \( \kappa > 10^3 \), for which the results become nearly independent of \( \kappa \) and streaming conductance tends to unity for any slipping area fraction \( a < 1 \). In Figs. 3(a) and 3(b), we show that this conclusion still holds for both stripe-alignments in a circular channel. When the EDL is not very thin, \( \kappa < O(10^3) \), or simply the uncharged regions become partial-slipping (dashed lines), finite inhibition effects on the EOF are evidently demonstrated in these figures. Figures 3(c) and 3(d) exhibit how the streaming conductance \( L_{12}^{||} \) and \( L_{12}^{\perp} \) will decrease with increasing \( a \). For a very thin EDL, \( \kappa = 1000 \), both streaming conductance coefficients are close to unity, i.e., the Helmholtz–Smoluchowski limit, until a sharp drop occurs upon approaching the limit at \( a = 1 \), where the coefficients become zero. When the EDL becomes thicker, say \( \kappa = 50 \), the streaming conductance departs from unity at a higher rate against \( a \), as has been pointed out by Ng and Chu [32]. Also, one should be aware of the dramatic effect by altering the period number \( K \) (for longitudinal-stripe-alignment) and the period length \( L \) (for transverse-stripe-alignment). A smaller number of stripes or a larger period length can reduce the rate of deviation from the nonslipping limit for the same slipping area fraction, implying a more abrupt drop on approaching \( a = 1 \). Conversely speaking, the slope close to \( a = 1 \) is milder for an increase of \( K \) or decrease of \( L \), which is in contrast to that in the first case for uniformly charged wall.

We further examine the effect of heterogeneous wall potentials. For walls with arbitrary slippage, the streaming conductance at the thin EDL limit is given in Eqs. (31) and (63). In particular, when the slipping and nonslipping regions are oppositely charged such that the potential of the slipping region is \( \zeta_S = -1 \), the expressions for the streaming conductance can be readily deduced as

\[
L_{12}^{||,\perp} = 1 - 2 \frac{\delta_{\parallel,\perp}}{\lambda} \kappa \delta_{\parallel,\perp} \quad \text{for } \zeta_S = -1, \quad \kappa \gg 1
\]

which is in agreement with that derived by Belyaev and Vinogradova [31]. They found that with oppositely charged interfaces, flow can be reversed on a surface of average zero or even positive charge, i.e., \( a_{\parallel,\perp} + (1 - a) \geq 0 \). Our numerical results in Figs. 4(a) and 4(b) exemplify this point. With wall slippage as modest as \( \lambda = 0.1 \), EOF opposite to the applied field direction (negative streaming conductance) can be induced with an average

![Fig. 3](https://example.com/fig3.png)

The longitudinal and transverse streaming conductance, \( L_{12}^{||} \) and \( L_{12}^{\perp} \), as functions of the Debye parameter \( \kappa \) and the slipping area fraction \( a \) where the intrinsic slip length \( \lambda = \infty \) and wall potential \( \zeta_S = 0 \). The number of periodic units for longitudinal stripes \( K = 5 \) in 3(a) and 3(c), while the period length for transverse stripes \( L = 1 \) in 3(b) and 3(d). Dashed lines in 3(a) and 3(b) represent the case where \( \lambda = 1 \). Dashed lines in 3(c) and 3(d) represent the case where \( K = 10 \) and \( L = 5 \) respectively.
We also see that the two lines approach the same limit when \( K \) is sufficiently large. It is because, as mentioned earlier, a larger \( K \) is equivalent to a smaller period length. When \( K \) is large enough, either intrinsic slip length presented here, \( \lambda = 0.1 \) or 1000, will become relatively larger than the period arc length, and therefore even the line \( \lambda = 0.1 \) will tend to that of \( \lambda = 1000 \) before they reach the limit of zero. Upon approaching this zero-limit, the negatively charged region can no longer provide slipping effect to the EOF and hence the streaming conductance will be the same independent of \( \lambda \) for a sufficiently large \( K \), sharing the limit of uniformly no-slip walls with heterogeneous wall potentials given in Eq. (33). In general, a smaller \( K \) is more capable of generating flow reversal. The same reasoning can be employed to explain the case with transverse stripes. As shown in the inset of Fig. 4(d), the effective slip length \( \delta_\perp \) is almost the same for all \( \lambda \) for a sufficiently small period length, \( L \leq 0.5 \). For larger \( L \), the lines branch out from each other and are saturated at \( L = O(10^2) \). When \( \lambda \) is sufficiently large, \( \lambda = 1000 \), it can be seen that the line approaches the asymptotic limit deduced by Lauga and Stone [20] when \( L \to \infty \) and \( a \) fixed.

\[
\delta_\parallel = \frac{2}{K} \ln \left[ \sec \left( \frac{\alpha \pi}{2} \right) \right] \quad \text{for} \quad \lambda = \infty \quad (68)
\]

\[
\delta_\perp \approx \frac{a}{4(1 - a)} \quad \text{for} \quad \lambda = \infty, \quad L \to \infty \quad (69)
\]

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\[
\delta_\parallel = \frac{2}{K} \ln \left[ \sec \left( \frac{\alpha \pi}{2} \right) \right] \quad \text{for} \quad \lambda = \infty \quad (68)
\]

\[
\delta_\perp \approx \frac{a}{4(1 - a)} \quad \text{for} \quad \lambda = \infty, \quad L \to \infty \quad (69)
\]

Similar to what we have observed in the case of longitudinal stripes, the streaming conductance will tend to the same limit as the intrinsic slip length becomes larger than the period length: \( \lambda > L \). This is best illustrated by the lines \( \lambda = 1000 \) and \( \lambda = 10 \) in the range \( L = O(1) \), within which they are almost the same. When the (dimensional) period length is much smaller than the radius of the tube, \( L \ll 1 \), the effective slip length is practically zero. Again, the negatively charged region can no longer provide any slippage and hence the streaming conductance will tend to the
limit given in Eq. (65), which is notably identical to Eq. (33) for longitudinal stripes as we have pointed out earlier. In either configuration, negatively charged regions with larger slippage is more likely to cause flow reversal.

The intriguing transition in convective patterns on an oppositely charged surface, as reported by Belyaev and Vinogradova [31], can also be found in the present problem. With the setting \( a = 0.35, \frac{\zeta_S}{\zeta_{NS}} = -0.43 \) and \( kL = 100 \), Belyaev and Vinogradova [31] showed that a slight increase of \( \lambda \) from 0.1 to 0.5 first brings forth additional convective patterns near no-slip regions, and further increase of \( \lambda \) to 100 may limit the recirculation to a zone near the no-slip regions when the flow is reversed. Here, we show in Fig. 5 that confinement by the tube may affect the formation of convective rolls and the resulting mixing mechanism. Although flow reversal is still possible for a slightly slipping oppositely charged region at \( \kappa = 100 \), we note that only with a very thin EDL, \( \kappa = 500 \), can such a full spectrum of transition in flow patterns be observed. On the other hand, when the EDL is considerably thick, \( \kappa = 10 \), no convective rolls can be formed.

The profile of a unidirectional EO flow is approximately maintained without appreciable variations in the axial direction. In summary, we find that the flow field is very sensitive to both the geometry as well as electrohydrodynamic properties of the wall. A slight change in these factors can significantly alter the morphology. Designers of microfluidic devices should therefore be mindful of these factors if they intend to exploit surface charge and slippage modulation as a means to flow adjustment.

**6 Concluding Remarks**

We have derived the effective equations for flow under pressure and electroosmotic forcings in a circular tube with periodic stripes of alternating hydrodynamic slip lengths and electric potentials. Flows with either longitudinal or transverse stripes are studied. Analytical limits for some particular kinds of stripe patterns are deduced which are analogous to those for electroosmotic flow (EOF) over an inhomogeneous surface or in a channel with non-uniform wall properties. The electrohydrodynamics in a circular

![Fig. 5 Streamlines of EOF transverse to the slipping stripes where slipping area fraction \( a = 0.35 \), period length \( \hat{L} = 1 \) and wall potential \( \zeta_w = -0.43 \). From left to right intrinsic slip length is increased \( \lambda = 0.001, 0.1, 100 \). From top to bottom Debye parameter is increased \( \kappa = 10, 100, 500 \). Dashed lines represent negative value of stream function, i.e., flow reversal.](image-url)
tube is qualitatively the same as that in other geometries. When the wall is uniformly charged, the effective slip length obtained from the hydrodynamic problem can be used directly in the EOF as if the wall were homogeneously slipping with slippage equal to the effective slip length. At the thin electric double layer (EDL) limit, the EOF is enhanced by a factor equal to the effective slip length multiplied by the Debye parameter. This enhancement factor can be significantly reduced by the presence of a small fraction of no-slip slots. We have here further found that such a reduction effect can be amplified with a shortened period length. In general, a shorter period length will give rise to a weaker slipping effect on the EOF.

This weakening effect can also be found for EOF over uncharged perfectly-slipping regions. A shorter period lowers the streaming conductance for the same slipping fraction. The EOF uncharged perfectly-slipping regions. A shorter period lowers the streaming conductance for the same slipping fraction. The EOF will also be reduced if the EDL is not thin enough, i.e., $\kappa < O(10^0)$). The thin EDL limit is practically achieved for $\kappa > O(10^0)$, where the EOF is unaffected by the uncharged slipping regions and the Helmholtz–Smoluchowski plug-flow velocity is attained. We have also confirmed that uncharged partial-slip regions always inhibit the EOF, whether at the thin EDL limit or not.

We have also looked into the scenario with oppositely charged slipping regions. Larger slippage at the oppositely charged regions as well as thinner EDL will facilitate flow reversal which can happen even when the wall is on the average electropositive. The convective patterns resembling those displayed by EOF on an anisotropic superhydrophobic surface have been reproduced. We, however, emphasize that this intriguing flow morphology is very sensitive to both the geometry and electrohydrodynamic properties of the wall.

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**Nomenclature**

- $a$ = slipping area fraction of the tube wall
- $E$ = electric field
- $E_{ext}$ = externally applied electric field
- $E_z$ = z-component of the externally applied electric field
- $E_{z\perp}$ = electric forcing parameter with dimensions of velocity
- $E_{z\parallel}$ = electric forcing parameter with dimensions of current density
- $e$ = elementary charge
- $I_z$ = electric current density in the $z$-direction
- $K$ = number of periodic units of longitudinal stripes on the tube wall
- $k_B$ = Boltzmann constant
- $L$ = half period length of one periodic unit of transverse stripes on the wall
- $L_{1\parallel\perp}$ = hydrodynamic conductance of the (longitudinal, transverse) flow
- $L_{1\perp\parallel}$ = streaming flow conductance of the (longitudinal, transverse) flow
- $L_{1\parallel\parallel}$ = streaming current conductance of the (longitudinal, transverse) flow
- $L_{1\perp\perp}$ = electrical conductance of the (longitudinal, transverse) flow
- $n_0$ = bulk concentration of ions at the neutral state
- $P_z$ = applied pressure gradient in the $z$-direction
- $P_{z\perp}$ = pressure forcing parameter with dimensions of velocity
- $P_{z\parallel}$ = pressure forcing parameter with dimensions of current density
- $p$ = induced pressure
- $p_{EO}$ = induced pressure in EOF
- $p_{EO}$ = electrostatic effective pressure
- $p_{PO}$ = induced pressure in Poiseuille flow

**Greek Symbols**

- $\delta_{z\parallel}$ = effective slip length of the (longitudinal, transverse) flow
- $\hat{\psi}$ = dielectric constant of the electrolyte
- $\hat{z}_{NS}$ = $\hat{z}$ potential of the nonslipping region
- $\hat{z}_S$ = $\hat{z}$ potential of the slipping region
- $\kappa$ = Debye parameter, inverse of Debye length
- $\lambda$ = slip length of the slipping region
- $\mu$ = dynamic viscosity of the fluid
- $\rho_e$ = charge density
- $\sigma$ = electric conductivity of the fluid
- $\psi$ = electric potential due to electric double layer
- $\psi_{tot}$ = total electric potential

**References**


